Planetary magnetospheres
Problems and **Solutions**

(Problems: August 2, 2010 Solutions: November 3, 2010 vmv)

1. *(Vol. I, 10.3.1)* Consider two extreme limits for the shape of a closed magnetosphere: [1] plane perpendicular to the planet-Sun line (the original version of the Chapman-Ferraro model), [2] sphere centered on the planet. (Note: [1] can be thought of as limiting case of a sphere of radius R centered at $R - R_{MP}$ behind the planet with $R \to \infty$.) For both cases:

   (a) Calculate the magnetic field everywhere inside the magnetosphere (under the assumption that interior sources of magnetic field except the planet's dipole are negligible, dipole axis is perpendicular to solar wind flow).

   (b) What is the ratio of field strength just inside the subsolar magnetopause to the dipole field there? What is the disturbance field at the planet compared to that at the subsolar magnetopause?

   (c) Show that the magnetic field in the equatorial plane is stronger than the dipole field everywhere (including the nightside). Show that every field line crossing the equatorial plane at a particular location reaches the planet at a higher latitude than the dipole field line from the same location. Can you generalize these results to less extreme shapes?

   (d) The location of the subsolar magnetopause is assumed fixed by pressure balance with the solar wind. For these special shapes, can pressure balance be satisfied anywhere else? if yes, where? if not, which way is the imbalance? Use the Newtonian approximation for the external pressure $\rho V^2(\hat{\mathbf{V}} \cdot \hat{\mathbf{n}})^2$.

1. **Solution:**

   (a) The magnetic field inside the magnetosphere can be represented as a superposition of the field of a dipole at the origin $\mathbf{r} = 0$ plus a disturbance field given by $-\nabla\Psi$ where the potential Ψ satisfies Laplace's equation $\nabla^2\Psi = 0$ everywhere inside the magnetosphere (including $\mathbf{r} = 0$) and obeys the boundary condition $\hat{\mathbf{n}} \cdot (-\nabla\Psi + \mathbf{B}_{dipole}) = 0$ at the magnetopause.
   [1]: The magnetopause is a plane perpendicular to the equatorial plane of the dipole and located a distance R_{MP} sunward. The magnetic field is obtained by adding the field of an image dipole of moment equal to (and aligned with) the magnetic moment μ of the planet, located an equal distance on the other side of the magnetopause plane (hence a distance $2R_{MP}$ sunward of the planet, in the equatorial plane).
   [2] The magnetopause is a sphere centered on the planet, of radius R_{MP}. The magnetic field is obtained by adding a uniform field $\mathbf{B}_0$ antiparallel to the dipole moment (parallel to the dipole field at the

equator); the boundary condition applied at the pole fixes the magnitude $B_0 = 2\mu/(R_{MP})^3$. (Can be derived by standard methods of potential theory, expressing Laplace's equation in spherical coordinates.)

(b) **Dipole field strength just inside the subsolar magnetopause $= \mu/(R_{MP})^3$. Disturbance field at the subsolar magnetopause $= \mu/(R_{MP})^3$ for [1] (image dipole at same distance as planet's dipole), $= 2\mu/(R_{MP})^3$ for [2]; hence ratio of total field to dipole field =2 for [1], =3 for [2]. Disturbance field at the planet (r $= 0$) is smaller than at the magnetopause by a factor 8 for [1] (image dipole twice as far away), the same as at the magnetopause for [2] (disturbance field uniform throughout the magnetosphere).**

(c) **For both [1] and [2], the disturbance field in the equatorial plane is everywhere in the same direction ($\hat{\mathbf{z}}$) as the dipole field, hence the total field is larger than the dipole field alone.**
Everywhere within the magnetosphere, the disturbance field is closer to the $\hat{\mathbf{z}}$ direction than is the dipole field (for [2], this is trivially obvious; for [1], note that at any point the latitude relative to the image dipole is lower than the latitude relative to the planet's dipole). When a field line of the total magnetic field is traced starting from a point in the equatorial plane, it is always displaced upward from where it would be in the dipole field alone.

(d) **For [1], everywhere at the magnetopause $\hat{\mathbf{V}} \cdot \hat{\mathbf{n}} = 1$ and the external (solar-wind) pressure is ρV^2. The internal (magnetic) pressure decreases away from the subsolar point. Pressure balance is thus satisfied at the subsolar point ONLY; everywhere else the external pressure is larger than the internal.**
For [2], $\hat{\mathbf{V}} \cdot \hat{\mathbf{n}} = \sin\theta\cos\phi$ and the external pressure is $\rho V^2 \sin^2\theta\cos^2\phi$ (spherical coordinates, $\theta = 0$ at the north pole, $\phi = 0$ at the sunward direction). The internal pressure at the magnetopause is proportional to $\sin^2\theta$. Pressure balance is satisfied on the entire noon meridian semicircle; elsewhere on the day side, the internal pressure is larger than the external. (Note that the Newtonian approximation is not applicable on the night side, where $\hat{\mathbf{V}} \cdot \hat{\mathbf{n}} < 0$.)

2. *(Vol. I, 10.3.2)* Consider a magnetotail cross-section with radius R_{MT}, lobe magnetic field B_{MT}, and plasma sheet of uniform thickness $Z_{PS} \ll R_{MT}$; assume magnetic field inside the plasma sheet decreases linearly with z from B_{MT} at $z = Z_{PS}$ to zero at $z = 0$; total pressure is constant over the cross-section.

   (a) Calculate the total force exerted across the cross-section (in which direction?), identifying explicitly the effects due to the presence of the plasma sheet.

   (b) Field lines threading the plasma sheet are generally assumed to be closed. Calculate the open and the closed contributions to the total magnetic flux across the cross-section.

(c) With 2b in mind, describe what exerts the force of 2a.

2. **Solution:**

(a) **The force is given by the surface integral of the stress tensor** *(Vol. I, equation 10.6)*; **with the normal to the cross-section taken as pointing antisunward, the integral represents the force exerted by the magnetotail on the volume containing the planet. With conventional magnetospheric coordinates ($\hat{\mathbf{x}}$ sunward, $\hat{\mathbf{z}}$ northward), $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$; $F_x > 0$ means sunward force, and in the present model $F_y = F_z = 0$. With total pressure (plasma + magnetic) assumed constant $= {B_{MT}}^2/8\pi$ over the cross-section, the contribution of the pressure terms to F_x is $\pi {R_{MT}}^2 {B_{MT}}^2/8\pi$. The contribution of the magnetic tension terms from the lobes is**

$$-({B_{MT}}^2/4\pi)(\pi {R_{MT}}^2 - 4Z_{PS}R_{MT})$$

and from the plasma sheet

$$-({B_{MT}}^2/4\pi)4R_{MT}\int_0^{Z_{PS}} dz(z/Z_{PS})^2 = -({B_{MT}}^2/4\pi)(4/3)Z_{PS}R_{MT} \ .$$

Adding everything together gives

$$F_x = -\frac{{B_{MT}}^2}{8\pi}\pi {R_{MT}}^2\left(1 - \frac{16}{3\pi}\frac{Z_{PS}}{R_{MT}}\right) \ ,$$

a net antisunward force.

(b) **Open flux (lobes):** $B_{MT}(\pi {R_{MT}}^2/2 - 2Z_{PS}R_{MT})$
Closed flux (plasma sheet): $2B_{MT}R_{MT}\int_0^{Z_{PS}} dz(z/Z_{PS}) = B_{MT}Z_{PS}R_{MT}$
(Note: magnetic flux is calculated from one half of the magnetotail cross-section, the force from the entire cross-section.)

(c) **The force from the lobes is exerted by tension of open field lines and comes ultimately from the pull of the flowing solar wind. The force from the plasma sheet is the tension of the stretched-out closed field lines, balanced by the decrease of the total pressure with increasing distance down the magnetotail.**

3. *(Vol. I, 10.3.3)* The total pressure over the cross-section of the magnetotail must be in balance with the external pressure from the solar wind. Typical B_{MT} values in the near-Earth magnetotail are $\sim 1/3$ of what would hold off the solar wind dynamic pressure ρV^2.

(a) Using the Newtonian approximation for the external pressure, estimate the implied flaring angle of the tail magnetopause.

(b) If the solar wind thermal and magnetic pressure is $\sim 10^{-2}\times$ dynamic pressure, and if the magnetic flux within the magnetotail remains constant (negligible reconnection), what would be the limiting value of R_{MT}? Can you estimate at how far down the tail that would be reached?

3. **Solution:**

   (a) **The flaring angle α is given by $\sin\alpha = \hat{\mathbf{V}} \cdot \hat{\mathbf{n}}$. Pressure balance of the magnetotail field with the external pressure given by the Newtonian approximation implies $\hat{\mathbf{V}} \cdot \hat{\mathbf{n}} = B_{MT}/\sqrt{8\pi\rho V^2}$; thus $\alpha \sim \sin^{-1} 0.33 \simeq 20°$.**

   (b) **The quoted value of B_{MT} implies that the external pressure outside the near-Earth magnetotail is $\sim 10^{-1}\times$ dynamic pressure, so the limiting value of R_{MT} is reached when B_{MT} decreases by another factor ~ 3. Constancy of flux implies that R_{MT} varies as $1/\sqrt{B_{MT}}$, hence the limiting value of R_{MT} is larger than the near-Earth value ($R_0 \sim 20R_E$) by a factor $\sim \sqrt{3}$, or $R_{MT\,lim} \sim 35R_E$.**

   The rate of increase of R_{MT} with distance x is related to the flaring angle:

   $$dR_{MT}/dx = \tan\alpha \simeq \sin\alpha \simeq (B_{MT}/\sqrt{8\pi\rho V^2})_0 (R_0/R_{MT})^2 \text{ , or}$$

   $$d{R_{MT}}^3/dx \simeq 3{R_0}^2 (B_{MT}/\sqrt{8\pi\rho V^2})_0$$

   (the subscript 0 refers to near-Earth values). This gives an estimate of the distance tailward of the Earth at which $R_{MT\,lim} \sim 3R_0$ is reached:

   $$x_{lim} \simeq x_0 + R_0(3\sqrt{3} - 1) \sim 94R_E \ .$$

   (A more accurate treatment, adding a thermal pressure term to the Newtonian approximation, gives $x_{lim} \sim 140R_E$ [Coroniti and Kennel 1972, JGR 77, 3361].)

4. *(Vol. I, 10.3.3)* Assuming the polar cap potential Φ_{PC} is some fraction f of the solar wind potential across a distance equal to the size of the dayside magnetosphere R_{MP} (defined by pressure balance), and representing the polar cap (= region of open field lines) as a circle of radius $R_E \sin\theta_{PC} \simeq R_E\theta_{PC}$, calculate the effective length $\mathcal{L}_T$ of the magnetotail. Compare the amount of open magnetic flux with the dipole flux beyond the distance R_{MP}. Describe the shape and size of the open field line region in the undisturbed solar wind (David Stern's "window").

4. **Solution:**

   An expression for $\mathcal{L}_T$ in terms of other parameters is given in *Vol. I, equation 10.18* [note misprint: R_P should be ${R_P}^2$, same in equation 10.17] **($\mathcal{L}_X \equiv fR_{MP}$). A more illuminating version is obtained if θ_{pc} is expressed in terms of the equivalent dipole L value $\sin^2\theta_{pc} = R_E/L$ and B_E in terms of the dipole moment $\mu = B_E{R_E}^3$:**

   $$\mathcal{L}_T \mathcal{L}_X B_s \approx 2\pi\mu/L$$

   ($B_s \equiv \mathbf{B}_{sw} \cdot \hat{\boldsymbol{\mu}}$). The right-hand side is the dipole flux beyond equatorial distance L. Typical observed $\theta_{pc} \approx 15°$ corresponds to $L \sim 15R_E$, somewhat larger than typical observed $R_{MP} \sim 10R_E$; thus the open flux can be nearly

comparable to ($\sim 30\%$ less than) the dipole flux beyond R_{MP}.

Dividing the above equation by ${R_{MP}}^3$, expressing $\mu/{R_{MP}}^3$ in terms of solar wind parameters through *Vol. I, equation 10.1*, **and solving for $\mathcal{L}_T$ gives**

$$\mathcal{L}_T \approx R_{MP} \frac{R_{MP}}{L} \frac{2\pi}{f\xi} \frac{\sqrt{8\pi\rho V^2}}{B_s}$$

from which it is obvious that in general $\mathcal{L}_T >> R_{MP} > fR_{MP} = \mathcal{L}_X$. The open field line region in the undisturbed solar wind, roughly a rectangle with dimensions $\mathcal{L}_X \times \mathcal{L}_T$, is thus greatly elongated in the direction of solar wind flow, having a width narrower than the magnetosphere and a length approximately equal to the entire magnetotail.

5. *(Vol. I, 10.4.1)* Idealize the region below the ionosphere as perfectly non-conducting, down to an assumed highly conducting region below the Earth's surface, with $\nabla^2\Phi = 0$. What are the electric fields in the region below the ionosphere that are produced as the result of magnetospheric convection and the associated ionospheric electric fields? (assume horizontal scale lengths are large compared to the vertical separation between the ionosphere and Earth's conducting layer). Specifically, at the Earth's surface what would be the maximum vertical **E** below the polar cap with $\Phi_{PC} = 50$ kV? At roughly what distance from a 12V DC power cable would one have a field of this magnitude? (It has been suggested that Φ_{PC} could be monitored by ground-level measurements of the vertical **E** on the polar ice caps. From the above estimate, would you expect this to be an easily practicable method?)

5. **Solution:**

Magnetospheric convection implies a horizontal variation of the electric potential in the ionosphere (see, e.g., *Vol. I, figure 10.5 left*, **where the streamlines are also equipotentials of the electric field). A highly conducting region below the Earth's surface implies a constant electric potential (in the frame of reference rotating with the Earth). There is thus a horizontally varying vertical difference of potential between the ionosphere and the Earth (in any frame of reference). The equation $\nabla^2\Phi = 0$ is to be solved with the boundary conditions $\Phi = \Phi_{IS}(\theta, \phi)$ at the top and $\Phi = \Phi_E = constant$ at the bottom. With the vertical separation h (~ 200 km) very small compared to an Earth radius and assumed small compared to the scale of horizontal variation of Φ_{IS}, the solution is**

$$\Phi \approx \frac{z}{h}\ (\Phi_{IS} - \Phi_E)$$

where z is the altitude above the conducting region. At any geographical location, there is a constant vertical electric field $E_z \approx -(\Phi_{IS} - \Phi_E)/h$ and a horizontal electric field that decreases with altitude from its ionospheric value to zero at the bottom.

With a polar cap potential of 50 kV**, the maximum vertical field is roughly** $50\ \mathrm{kV}/100\ \mathrm{km} = 0.5\ \mathrm{V\ m}^{-1}$**. An unshielded 12V DC power cable, laid out in a straight line (far from other conductors), would produce an electric field of magnitude larger than this out to distances of 24 meters. Measuring the vertical electric field of ionospheric origin might thus require some care with stray fields from other sources.**

6. *(Vol. I, 10.4.1)*

   (a) Electrons and positive ions in equal concentration (number density) n are injected with zero initial velocity into crossed **E** and **B** fields. The particles follow cycloidal trajectories, curving in opposite sense for + and - charges, but both have mean drift velocity $\mathbf{V} = c\mathbf{E} \times \mathbf{B}/B^2$.

   (b) The same particles are injected with initial velocity **V** (same for + and - charges) perpendicular to the magnetic field, with zero electric field. The particles gyrate in opposite sense for + and - charges.

   In both cases, the cycloidal/gyro motion produces charge separation. Assuming the medium is locally homogeneous, calculate the implied electric field $\delta\mathbf{E}$. Show that it is opposite to the applied **E** for case 6a and in the direction of $-\mathbf{V} \times \mathbf{B}$ for case 6b. For what value of n does $\delta\mathbf{E}$ become comparable to (-) applied **E** (case 6a) or to $-\mathbf{V} \times \mathbf{B}/c$ (case 6b)? (express as a dimensionless ratio involving n).

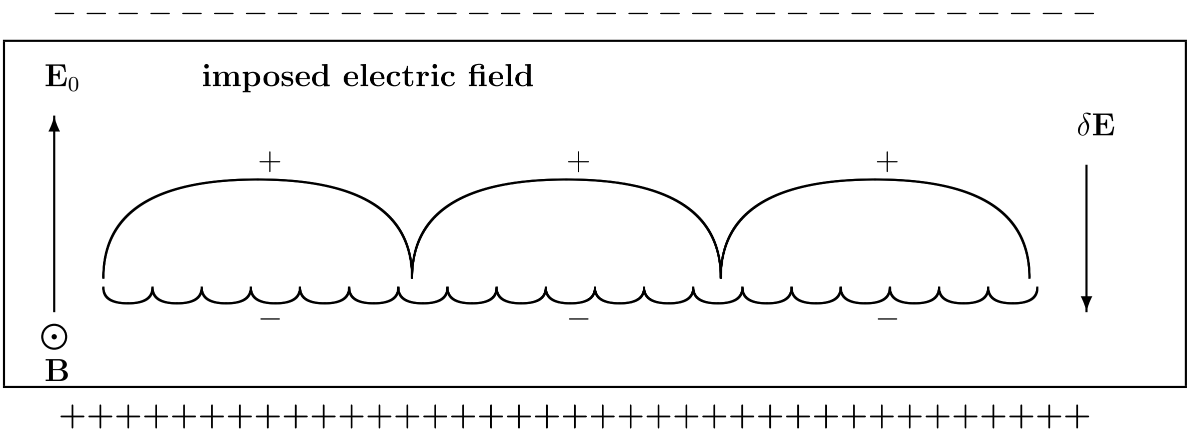
6. **Solution:**

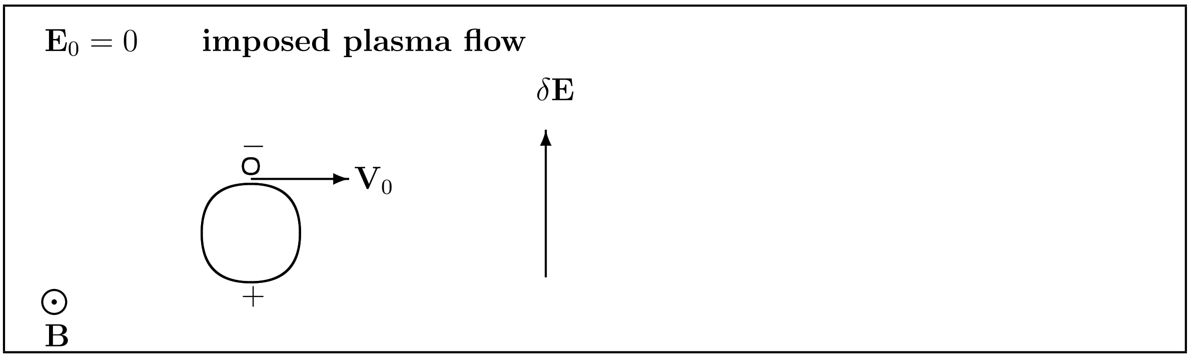
   Each injected particle is displaced by the gyromotion in a direction perpendicular to the velocity; the displacements of positive and negative particles are in opposite directions, and the average distance of displacement is equal to the gyroradius based on the mean drift speed (case 6a) or on the injection speed (case 6b). At the boundaries of the locally homogeneous region, this produces an excess charge layer (+ on one side, - on the other), with surface charge density enr_g**, where** r_g **is the gyroradius of the positive or negative particles. The electric field of these charge layers is given by**

$$\delta\mathbf{E} = -4\pi n\, e \left[\frac{c\mathbf{E}}{B}\right] \left[\frac{(m_i + m_e)c}{eB}\right] \qquad \textbf{(case 6a)}$$

$$\delta\mathbf{E} = -4\pi n\, e \left[\mathbf{V} \times \frac{\mathbf{B}}{B}\right] \left[\frac{(m_i + m_e)c}{eB}\right] \qquad \textbf{(case 6b)}\ .$$

   The direction of δ**E is most easily determined by simply drawing the appropriate pictures (the charges ++++ and − − −− outside the boxes are those of the imposed E, and inside the boxes are those of** δ**E):**





From the pictures it is obvious that the direction of $\delta\mathbf{E}$ is opposite to that of the imposed E in case 6a and aligned with that of $-\mathbf{V}\times\mathbf{B}/c$ in case 6b. The magnitude of $\delta\mathbf{E}$ is proportional to that of the imposed quantity, the constant of proportionality being in both cases

$$\frac{4\pi n(m_i + m_e)c^2}{B^2} = \frac{c^2}{{V_A}^2} \ .$$

If n is sufficiently large so that the Alfvén speed is less than the speed of light, $\delta\mathbf{E}$ becomes dominant; in the equation above for case 6a, one must now write $\mathbf{E} = \mathbf{E}_0 - \delta\mathbf{E}$, where $\mathbf{E}_0$ is the externally imposed field.

7. *(Vol. I, 10.4.2)* Consider a rotating sphere with an imbedded magnetic dipole (moment aligned with the rotation axis). The sphere is assumed to be highly conducting (it is sufficient to assume only that the outer layers are highly conducting — why?). Calculate the electric potential as function of latitude and longitude on the surface. From that, calculate the electric field everywhere outside the sphere, under the assumption that

   (a) the sphere is surrounded by a vacuum,

   (b) the sphere is surrounded by a plasma, sufficiently dense to enforce the MHD approximation, but sufficiently tenuous so all mechanical stresses are negligible.

Show that only in case 7b does the $\mathbf{E} \times \mathbf{B}$ drift correspond to corotation.

7. **Solution:**

Within the highly conducting sphere, the electric field in the corotating frame is zero; in the inertial frame, therefore, $\mathbf{E} = -\mathbf{V} \times \mathbf{B}/c$ with V the corotation velocity. Just below the surface of the sphere at $r = R_P$, the tangential electric field is

$$E_\theta = -V_\phi B_r/c = -\left(2B_P R_P \Omega/c\right) \sin\theta \cos\theta$$

from which the potential at the surface can be obtained by integrating with respect to θ. (What happens much below the surface makes no difference.)

(a) **If the sphere is surrounded by vacuum, the electric potential outside is given by the solution of Laplace's equation with boundary conditions of zero at infinity and specified potential at the surface of the sphere. It is convenient to integrate E_θ by setting $\sin\theta d\theta = -d\ \cos\theta$ and to write the potential at the surface as**

$$\Phi = -\frac{B_P {R_P}^2 \Omega}{c} \cos^2\theta + constant = -\frac{2}{3}\frac{B_P {R_P}^2\Omega}{c}\left[\frac{3}{2}\cos^2\theta - 1\right] \, .$$

The expression in [ ] is the Legendre polynomial of order 2, which divided by r^3 is a solution of Laplace's equation (axially symmetric quadrupole field). The potential in space is therefore

$$\Phi = -\frac{B_P {R_P}^2\Omega}{c}\frac{{R_P}^3}{r^3}\left[\cos^2\theta - \frac{2}{3}\right] \, .$$

To show that the $\mathbf{E} \times \mathbf{B}$ drift from this field does NOT correspond to corotation, the tedious algebra of computing the fields and the cross product is not necessary; it suffices to note that E varies as $\sim r^{-4}$ and B as $\sim r^{-3}$, hence E/B varies as $\sim r^{-1}$ (not as $\sim r$ of corotation velocity).

(b) **In this case, magnetic field lines are equipotentials of the electric field, and the potential is determined by mapping from the surface along dipolar field lines. It is now convenient to integrate E_θ by setting $\cos\theta d\theta = d\ \sin\theta$ and to write the potential at the surface as**

$$\Phi = \frac{B_P {R_P}^2\Omega}{c}\sin^2\theta = \frac{B_P {R_P}^3\Omega}{c\ L}$$

where $L = r/\sin^2\theta$ is constant along a dipole field line; the second expression therefore gives Φ in space as well. To compute the $\mathbf{E} \times \mathbf{B}$

drift it is useful to remember that the Euler potentials of the dipole magnetic field are μ/L and ϕ:

$$\mathbf{B}_{dipole} = \nabla\left(\frac{B_P {R_P}^3}{L}\right) \times \nabla\phi = \nabla\left(\frac{B_P {R_P}^3}{L}\right) \times \frac{\hat{\phi}}{r\sin\theta}\ .$$

With E and B both containing the factor $\nabla(1/L)$, it is simple to show that $c\mathbf{E}\times\mathbf{B}/B^2 = \hat{\phi}\,\Omega r\sin\theta$.

8. *(Vol. I, 10.4.2)* For rigid corotation of plasma in a dipole magnetic field (dipole aligned with rotation axis),

   (a) calculate the charge density ρ_c implied by $\nabla\cdot\mathbf{E}$ (in inertial frame of reference). How does ρ_c vary with location along a particular field line? Calculate numerical values in electron charges per cm^3 for some typical location in the magnetospheres of Earth, Jupiter, Saturn; compare with typical observed electron/ion concentrations.

   (b) Now consider the same situation from the corotating frame of reference. In this frame, the electric field is zero and therefore also $\nabla\cdot\mathbf{E} = 0$. Verify that ρ_c in the corotating frame is almost the same as in the inertial frame and therefore non-zero (calculate the relativistic transformation corrections and show they are small). What does this imply about Maxwell's equations?

8. **Solution:**

   (a) If $\mathbf{E} = -\mathbf{V}\times\mathbf{B}/c$, then $\nabla\cdot\mathbf{E} = -\mathbf{B}\cdot\nabla\times\mathbf{V}/c + \mathbf{V}\cdot\nabla\times\mathbf{B}/c$. For corotation, $\nabla\times\mathbf{V} = 2\boldsymbol{\Omega}$. (The $\nabla\times\mathbf{B} = (4\pi/c)\mathbf{J}$ term is negligible, by the assumptions stated in problem 7b.) The charge density is given by

$$4\pi\rho_c \equiv 4\pi\ e\ \Delta n = -\ \frac{2\boldsymbol{\Omega}\cdot\mathbf{B}}{c} = \frac{2\Omega\mu}{c\ r^3}\left(2 - 3\sin^2\theta\right) = \frac{2\Omega\mu}{c\ L^3}\ \frac{2-3\sin^2\theta}{\sin^6\theta}$$

   where the two expressions on the right are for a dipole field with moment antiparallel to the rotation axis (as at Earth).
   The charge density varies strongly along the field line and changes sign at $\theta = \sin^{-1}\sqrt{2/3} = 54.7°$ (latitude $35.3°$).

   The numerical value of Δn (the difference between + and - concentrations) can be obtained from $4\pi e\ \Delta n = \Omega B_z/c$. It is convenient to multiply both sides by e/m_e , converting the equation to ${\omega_p}^2 = 2\Omega\Omega_e$, where ω_p is the plasma frequency for an electron concentration Δn and Ω_e is the electron gyrofrequency. With

$$\omega_p/2\pi = 8.98\ \text{kHz}\ \sqrt{\Delta n/1\ \text{cm}^{-3}} \quad \Omega_e/2\pi = 2.80\ \text{MHz}\ B/1\ \text{G} \quad \Omega/2\pi = 1/\tau_{rot}$$

   we have $\Delta n = 8.0\times10^{-7}\ \text{cm}^{-3}(B/10^5\ \text{nT})(24\ \text{h}/\tau_{rot})$ (note that Δn depends only on magnetic field strength, rotation period, and atomic

parameters).

Some sample values (calculated Δn and observed n_e in cm^{-3}, B in nT) for different planetary magnetospheres (see also *Vol. I, chapter 13* **for parameter values):**

Earth ($\tau_{rot} = 24$ h)**:**
r$= 4\ R_E$**:** $B = 484$**,** $\Delta n = 3.9 \times 10^{-4}$
magnetotail: $B \approx 20$**,** $\Delta n = 1.6 \times 10^{-5}$
observed n_e**: from** $\sim 10^{-1}$ **to** ~ 4000

Jupiter ($\tau_{rot} = 9.8$ h)**:**
r$= 5.9\ R_J$ **(distance of Io):** $B = 2083$**,** $\Delta n = 4.1 \times 10^{-3}$
magnetotail: $B \approx 3$**,** $\Delta n = 6.7 \times 10^{-6}$
observed n_e**: from** $\sim 10^{-2}$ **to** ~ 3000

Saturn ($\tau_{rot} \simeq 10.5$ h)**:**
r$= 3.94\ R_S$ **(distance of Enceladus):** $B = 350$**,** $\Delta n = 6.4 \times 10^{-4}$
magnetotail: $B \approx 2$**,** $\Delta n = 3.7 \times 10^{-6}$
observed n_e**: from** $\sim< 10^{-1}$ **to** ~ 100

(b) **The Lorentz transformations of charge and current density are** $\rho_c' = \gamma(\rho - \mathbf{J} \cdot \mathbf{V}/c^2)$, $\quad \mathbf{J}' = \gamma(\mathbf{J} - \rho_c \mathbf{V})$. **In this problem we neglect all currents except those that are result from advection of charge density; thus** $\mathbf{J} \sim \rho_c \mathbf{V}$ **may occur in one frame or another. The resulting corrections to** ρ_c' **are of order** V^2/c^2**, negligible in most systems. The implication is that only in inertial frames of reference is the validity of Maxwell's equations in their usual form assured; the divergence equations in particular must be modified when used in a rotating frame.**

9. *(Vol. I, 10.4.3)* Construct a simple analytical model of magnetospheric convection by solving for the ionospheric potential Φ with $\nabla \Phi = -\mathbf{E}^*$ (corotating frame) or $\nabla \Phi = -\mathbf{E}$ (inertial frame) from equations (10.14) and (10.15) of Vol. I combined with $\nabla \cdot \mathbf{I}_\perp = J_\parallel$, under the following simplifying assumptions:

   (a) The polar cap is a circle at colatitude $\theta = \theta_{PC} \ll 1$.

   (b) The magnetic field is everywhere vertical; the Pedersen and Hall conductance values are the same everywhere.

   (c) The Birkeland (magnetic-field-aligned) current into or out of the ionosphere is zero except at the polar cap boundary (and possibly at other boundaries to be introduced later).

   (d) The potential at the polar polar cap boundary is specified as $\Phi_0 \sin \phi$, where ϕ is the local time measured from noon. (Hint: from the assumptions so far, show that all local-time dependence must be a combination of $\sin \phi$ and $\cos \phi$.)

(e) The equations may be solved in the flat-Earth approximation, $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$; however, the flat Earth is not allowed to be infinite, so the solution stops at the equator, and you have to decide what is the boundary condition there (hint: 9c is relevant). [For the mathematically more sophisticated: it is not that much more difficult to solve analytically without the flat-Earth approximation.]

Obtain expressions for the potential, electric field, and ionospheric current, as functions of θ and ϕ, and for Birkeland current as function of ϕ at $\theta = \theta_{PC}$.

Now introduce complete shielding of convection from $\theta > \theta_S$. Solve as above, with boundary condition $\Phi = 0$ at $\theta = \theta_S$ (what would be the implication of the boundary condition $\Phi = \text{constant} \neq 0$?) Show that there must be a Birkeland current at $\theta = \theta_S$; calculate its functional dependence on ϕ.

Can you suggest an association of the model Birkeland currents at $\theta = \theta_{PC}$ and at $\theta = \theta_S$ with anything that is observed?

9. **Solution:**

Equations (see *Vo. I, equations 10.8, 10.10, 10.14, 10.15***), written for Earth's northern hemisphere ($\hat{\mathbf{B}} = -\hat{\mathbf{r}}$), vertical B and constant conductance approximations:**

$$\mathbf{E} = -\nabla\Phi \tag{1}$$

$$\mathbf{I} = \Sigma_P \mathbf{E} + \Sigma_H \hat{\mathbf{r}} \times \mathbf{E} \tag{2}$$

$$\nabla \cdot \mathbf{I} = \Sigma_P \nabla \cdot \mathbf{E} = -\Sigma_P \nabla^2 \Phi = J_{\|} \tag{3}$$

(E in these equations is really $\mathbf{E}^*$. The difference between Φ in the corotating and in the inertial frame is the corotational potential of problem 7b.) The Birkeland current $J_{\|}$ is assumed zero except at the polar cap boundary $\theta = \theta_{PC}$ and possibly at other boundaries to be specified. Except at these boundaries, Φ satisfies Laplace's equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{4}$$

written in spherical coordinates with the radial derivatives omitted (Φ is a function of θ and ϕ only). The ϕ-dependence comes only from the boundary condition

$$\Phi = \Phi_0 \sin\phi \quad \text{at } \theta = \theta_{PC} \ . \tag{5}$$

Since all the equations are linear, a Fourier analysis in ϕ can be done, keeping only terms in $\sin\phi$ and $\cos\phi$, for which equation (4) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\tilde{\Phi}}{\partial\theta} - \frac{\tilde{\Phi}}{\sin^2\theta} = 0 \tag{6}$$

where $\tilde{\Phi}(\theta)$ is now the (complex) Fourier transform. (Additional terms with no ϕ-dependence are possible.)

It is sufficiently instructive to treat the problem in the flat-Earth approximation, $\sin\theta \simeq \theta$; then $\sin\theta \longrightarrow \theta$ everywhere in equation (6), and the two independent solutions are easily shown to be

$$\tilde{\Phi}_1 \sim \theta \qquad\qquad \tilde{\Phi}_2 \sim \frac{1}{\theta}$$

[The full spherical solutions, without the flat-Earth approximation, can be shown to be

$$\tilde{\Phi}_1 \sim \frac{2\sin\theta}{1+\cos\theta} \qquad\qquad \tilde{\Phi}_2 \sim \frac{1+\cos\theta}{2\sin\theta} = \frac{\sin\theta}{2(1-\cos\theta)}\ ].$$

Separate solutions apply for $\theta >$ or $< \theta_{PC}$, to be matched at this boundary. For $0 < \theta < \theta_{PC}$ (polar cap), only $\tilde{\Phi}_1$ applies (non-singular at $\theta = 0$), hence

$$\tilde{\Phi} = \Phi_0\,\frac{\theta}{\theta_{PC}} \qquad \text{or} \qquad \Phi = \Phi_0\,\frac{\theta}{\theta_{PC}}\sin\phi \tag{7}$$

For $\theta_{PC} < \theta < \theta_{eq}$,

$$\tilde{\Phi} = c_1\,\frac{\theta}{\theta_{PC}} + c_2\,\frac{\theta_{PC}}{\theta} \tag{8}$$

where c_1,c_2 are constants determined by two boundary conditions: (5) plus the boundary condition at the "equator" $\theta = \theta_{eq}$, which must be a condition of zero current flow normal to the equator (by symmetry, non-zero current would flow into the boundary from both sides). From equation (2)

$$-R_E I_\theta = \Sigma_P\,\frac{\partial\Phi}{\partial\theta} - \Sigma_H\,\frac{1}{\theta}\,\frac{\partial\Phi}{\partial\phi} \tag{9}$$

or Fourier-analyzed

$$-R_E\tilde{I}_\theta = \Sigma_P\,\frac{\partial\tilde{\Phi}}{\partial\theta} - i\Sigma_H\,\frac{\tilde{\Phi}}{\theta}$$

so the boundary condition is $I_\theta = 0$ at $\theta = \theta_{eq}$. The equations for c_1,c_2 from the two boundary conditions are

$$c_1 + c_2 = \Phi_0 \qquad\qquad \Sigma_P\left[c_1 - c_2\left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\right] - i\,\Sigma_H\left[c_1 + c_2\left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\right] = 0$$

which give

$$\begin{aligned} c_2 &= \Phi_0\left[1 + \left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\,\frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H}\right]^{-1} \\ c_1 &= \Phi_0\left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\,\frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H}\left[1 + \left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\,\frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H}\right]^{-1} \end{aligned}$$

and the solution

$$\tilde{\Phi} = \Phi_0 \, \left(\frac{\theta_{PC}}{\theta} + \frac{\theta \, \theta_{PC}}{{\theta_{eq}}^2} \, \frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H} \right) \left[1 + \left(\frac{\theta_{PC}}{\theta_{eq}} \right)^2 \, \frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H} \right]^{-1} \quad . \qquad (10)$$

The Fourier-analyzed potential is complex if $\Sigma_H \neq 0$; with Φ_0 taken as real, the real part of $\tilde{\Phi}$ is the coefficient of $\sin\phi$ and the imaginary part the coefficient of $\cos\phi$. Despite the seeming relative simplicity of equation (10), the algebra required to separate the real and imaginary parts in the general case is too tedious to be worthwhile, and two simplified special cases suffice. One is $\Sigma_H = 0$; $\tilde{\Phi}$ is then real, and

$$\Phi = \Phi_0 \, \sin\phi \, \left(\frac{\theta_{PC}}{\theta} + \frac{\theta \, \theta_{PC}}{{\theta_{eq}}^2} \right) \left[1 + \left(\frac{\theta_{PC}}{\theta_{eq}} \right)^2 \right]^{-1} \quad . \qquad (11)$$

The other is the (fairly realistic) limit $\theta_{PC} << \theta_{eq}$; expanded to lowest order, (10) becomes

$$\tilde{\Phi} = \Phi_0 \, \left[\frac{\theta_{PC}}{\theta} + \left(\frac{\theta_{PC}}{\theta_{eq}} \right)^2 \, \frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H} \left(\frac{\theta}{\theta_{PC}} - \frac{\theta_{PC}}{\theta} \right) \right]$$

and noting that

$$\frac{\Sigma_P + i\Sigma_H}{\Sigma_P - i\Sigma_H} = \frac{{\Sigma_P}^2 - {\Sigma_H}^2 + 2i\Sigma_P\Sigma_H}{{\Sigma_P}^2 + {\Sigma_H}^2}$$

we obtain

$$\begin{aligned} \Phi \quad &= \quad \Phi_0 \, \left[\frac{\theta_{PC}}{\theta} + \left(\frac{\theta_{PC}}{\theta_{eq}} \right)^2 \, \frac{{\Sigma_P}^2 - {\Sigma_H}^2}{{\Sigma_P}^2 + {\Sigma_H}^2} \left(\frac{\theta}{\theta_{PC}} - \frac{\theta_{PC}}{\theta} \right) \right] \sin\phi \\ &\quad + \; \Phi_0 \, \left[\left(\frac{\theta_{PC}}{\theta_{eq}} \right)^2 \, \frac{2\Sigma_P\Sigma_H}{{\Sigma_P}^2 + {\Sigma_H}^2} \left(\frac{\theta}{\theta_{PC}} - \frac{\theta_{PC}}{\theta} \right) \right] \cos\phi \; . \qquad (12) \end{aligned}$$

Much simpler is the situation for complete shielding at $\theta = \theta_S$. Equation (8) then holds for $\theta_{PC} < \theta < \theta_S$, and the second boundary condition is $\Phi = 0$ at $\theta = \theta_S$. The equations for c_1,c_2 now are

$$c_1 + c_2 = \Phi_0 \qquad\qquad c_1 \, \frac{\theta_S}{\theta_{PC}} + c_2 \, \frac{\theta_{PC}}{\theta_S} = 0$$

and the solution is easily obtained:

$$\Phi = \Phi_0 \, \sin\phi \, \left(\frac{\theta_{PC}}{\theta} - \frac{\theta \, \theta_{PC}}{{\theta_S}^2} \right) \left[1 - \left(\frac{\theta_{PC}}{\theta_S} \right)^2 \right]^{-1} \quad ; \qquad (13)$$

for $\theta_S < \theta < \theta_{eq}$, $\Phi = 0$. (If the boundary condition at $\theta = \theta_S$ had been taken as $\Phi = constant \neq 0$, a ϕ-independent solution of Laplace's equation

would be added to (13), representing a physically unrealistic azimuthal but non-corotational flow.)

From the solution for $\Phi(\theta, \phi)$, E and I are obtained by simple differentiation, equations (1) and (2). The Birkeland current $J_{\|}$ is assumed to be a δ-function at $\theta = \theta_{PC}$; the current per unit length $I_{\|}$ can be calculated from the discontinuity of I_θ and hence of $\partial\Phi/\partial\theta$:

$$I_{\|} = -\frac{\Sigma_P}{R_E}\left[\frac{\partial\Phi}{\partial\theta}\right]_{\theta_{PC}-}^{\theta_{PC}+} = \frac{\Sigma_P}{R_E}\frac{1}{\theta_{PC}}(\Phi_0 - c_1 + c_2) = \frac{2c_2}{\theta_{PC}}\frac{\Sigma_P}{R_E} .$$

From equation (11) ($\Sigma_H = 0$),

$$I_{\|} = \frac{2\Sigma_P\Phi_0}{R_E\,\theta_{PC}}\sin\phi\left[1 + \left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\right]^{-1} .$$

From equation (12) ($\theta_{PC} << \theta_{eq}$),

$$I_{\|} = \frac{2\Sigma_P\Phi_0}{R_E\,\theta_{PC}}\left\{\left[1 - \left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\frac{{\Sigma_P}^2 - {\Sigma_H}^2}{{\Sigma_P}^2 + {\Sigma_H}^2}\right]\sin\phi + \left(\frac{\theta_{PC}}{\theta_{eq}}\right)^2\frac{2\Sigma_P\Sigma_H}{{\Sigma_P}^2 + {\Sigma_H}^2}\cos\phi\right\} .$$

From equation (13) (shielding),

$$I_{\|} = \frac{2\Sigma_P\Phi_0}{R_E\,\theta_{PC}}\sin\phi\left[1 - \left(\frac{\theta_{PC}}{\theta_S}\right)^2\right]^{-1} .$$

In the case of shielding, there is in addition a Birkeland current at $\theta = \theta_S$, given by

$$I_{\|} = -\frac{\Sigma_P}{R_E}\left[\frac{\partial\Phi}{\partial\theta}\right]_{\theta_S-}^{\theta_S+} = \frac{\Sigma_P}{R_E}\frac{\partial\Phi}{\partial\theta}\bigg|_{\theta=\theta_S}$$

and from equation (13)

$$I_{\|} = \frac{2\Sigma_P\Phi_0}{R_E}\frac{\theta_{PC}}{{\theta_S}^2}\sin\phi\left[1 - \left(\frac{\theta_{PC}}{\theta_S}\right)^2\right]^{-1} .$$

The Birkeland currents at $\theta = \theta_{PC}$ and $\theta = \theta_S$ are generally identified with the observed Region 1 and Region 2 currents, respectively.

10. *(Vol. II, 10.2)* The component of the interplanetary magnetic field parallel to the planetary dipole moment (i.e., antiparallel to the dipole magnetic field in the equatorial plane) is generally considered the most effective in producing magnetospheric disturbances. If B_z southward (at Earth) is increased by a factor 2, the B_z southward may be thought roughly twice as much "geoeffective." However, if the 2× increase of B_z southward is accompanied by a 4× increase of solar wind density and no change of solar wind speed, the net change of "geoeffectiveness" is likely to be small. Can you think of a simple physical argument why this is so?

10. **Solution:**

The "geoeffectiveness" depends on the value of B_z southward not by itself but in relation to the strength of the geomagnetic field in the outermost magnetosphere. Everything else being equal, however, the strength of the geomagnetic field just inside the magnetopause, should scale as the square root of the solar wind dynamic pressure. An increase of B_z southward accompanied by an increase of solar wind density, keeping the ratio $B_z/\sqrt{8\pi\rho V^2}$ unchanged, should therefore not change "geoeffectiveness" much.

11. *(Vol. II, 10.6.2)* Assume the magnetosphere is filled, out to a radial distance R_{MP} ($\equiv$ distance to subsolar magnetopause), with plasma having a uniform isotropic pressure equal in value to the solar wind dynamic pressure ρV^2. Calculate the disturbance field depression at Earth predicted by the Dessler-Parker-Sckopke formula (equation (10.18) of Vol. II) and show that it is close in magnitude to the field compression predicted from the dayside magnetopause currents. Is this a coincidence, or is there a physical effect behind it? To be classed as intense, a magnetic storm must have maximum Dst depression much larger than this; what (if any) are the implications?

11. **Solution:**

The energy density of isotropic pressure is $(3/2)P$. With the stated assumptions, the Dessler-Parker-Sckopke formula gives

$$\mu b(0) = (4\pi/3)\,(R_{MP})^3\,3\rho V^2\ .$$

From *Vol. I, equation 10.1*

$$\mu/\,(R_{MP})^3 = \sqrt{8\pi\rho V^2}/\xi$$

and combining the above two equations

$$b(0) = (\xi/2)\,\sqrt{8\pi\rho V^2}/\xi = \left(\xi^2/2\right)\mu/\,(R_{MP})^3$$

which is roughly of the same order of magnitude as the disturbance field from magnetopause currents (see solution to problem 1). The two disturbance fields, however, have opposite signs and would tend to cancel each other. The physical reason is that both the magnetopause current and the ring current result from pressure gradients; if the plasma pressure in the magnetosphere were uniform and equal to the external pressure (idealized as also uniform), there would be no pressure gradients and hence no disturbance fields.

A geomagnetic storm is in essence the inflation of the geomagnetic field by enhanced internal pressure. For a storm to be classed intense, the inflation should significantly exceed the opposite effect of compression by external pressure.

12. *(Vol. I, 10.6)* Write down the dimensionally correct form for the dependence of the polar cap potential Φ_{PC} on solar wind parameters. There is observational evidence ("transpolar potential saturation") that for large values of VB_s in the solar wind ($B_s \equiv B_z$ southward), Φ_{PC} approaches a value independent of VB_s. Can you find a dimensionally correct formula for Φ_{PC} that is independent of VB_s? if yes, how does it depend on the other parameters?

12. **Solution:**

The general dimensionally correct formula for Φ_{PC} (from *Vol. I, equation 10.39 and following discussion***):**

$$c\Phi_{PC} \sim V\ B\ R_{MP}\ \Psi\left(\frac{B}{\sqrt{4\pi\rho V^2}}\,,\frac{4\pi\Sigma_P V}{c^2}\right)$$

where the subscript sw on solar-wind parameters ρ, V, B has been omitted and Ψ is a function of two dimensionless quantities, written out in full. To obtain a form for Φ_{PC} that is independent of the product vB (but possibly depending on V through the dynamic pressure ρV^2), Ψ must vary as the inverse of both variables; then

$$c\Phi_{PC} \sim \left[\frac{c^2}{4\pi\Sigma_P}\right]\left[\sqrt{4\pi\rho V^2}\right]\ R_{MP}$$

where the two quantities in [ ] have the dimensions (in Gaussian units) of V and B, respectively. R_{MP} depends on solar-wind dynamic pressure as $\sim (\rho V^2)^{-1/6}$. The limiting value of Φ_{PC} depends on dynamic pressure as $\sim (\rho V^2)^{1/3}$ and on ionospheric Pedersen conductance as $\sim {\Sigma_P}^{-1}$. [Note a misprint in the discussion of this topic in Vol. I, 10.64: in equation 10.56, denominator should be $1 + Q_1 Q_2 F$.]

Shocks, Homework

13. a. The hydrodynamic jump conditions for plasma, $\gamma = 5/3$, are

$$M_2^2 \;=\; \frac{M_1^2+3}{5M_1^2-1} \;\rightarrow\; \frac{1}{5} \tag{1}$$

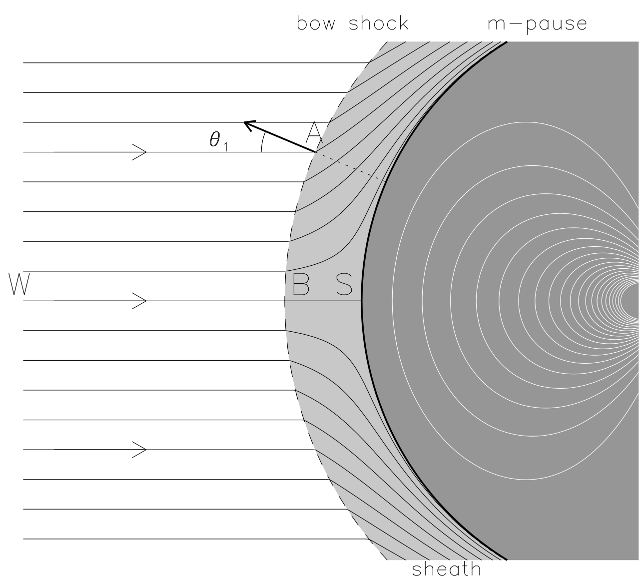
$$\frac{\rho_2}{\rho_1} = \frac{v_{n,1}}{v_{n,2}} \;=\; \frac{4}{1+3/M_1^2} \;\rightarrow\; 4 \quad , \tag{2}$$

where limits are for $M_1 \rightarrow \infty$. We can use these to write

$$\frac{p_2}{\rho_2} \;=\; \frac{v_2^2}{\gamma M_2^2} \;\rightarrow\; 3\,v_2^2 \;\rightarrow\; \frac{3}{16}\,v_1^2 \;=\; \frac{3}{16}\,v_{sw}^2 \quad . \tag{3}$$

This gives the temperature

$$T_B \;=\; \frac{m_p}{2k_{\rm B}}\frac{p_B}{\rho_B} \;\simeq\; \frac{3m_p}{32k_{\rm B}}\,v_{sw}^2 \quad . \tag{4}$$



A simplified of the day-side magnetosphere. Flow streamlines are shown as solid curves originating at the Sun, far to the left. The bow shock is a dashed arc and the magnetopause is a thick solid arc. The magnetosphere proper is dark grey, with some white magnetic field lines shown. The magnetosheath is the lighter greay region between the bow shock and the magnetopause.

b. Using eq. (3) gives

$$\tfrac{1}{2}v_B^2 \;=\; \frac{1}{6}\frac{p_B}{\rho_B} \quad , \tag{5}$$

and thus

$$\frac{p_B}{\rho_B} \;+\; \frac{2}{5}\,\tfrac{1}{2}v_B^2 \;=\; \frac{16}{15}\frac{p_B}{\rho_B} \;=\; \frac{p_S}{\rho_S} \quad . \tag{6}$$

From this we find

$$T_S \;=\; \frac{16}{15}T_B \;=\; \frac{m_p}{10k_{\rm B}}\,v_{sw}^2 \;=\; 1.2\times10^5\,{\rm K}\left(\frac{v_{sw}}{100\,{\rm km/s}}\right)^2 \quad . \tag{7}$$

Taking $v_{sw} = 800$ km/s gives $T_S = 7.7\times10^6$ K.

c. It is clear from eq. (2) that

$$\rho_B \;=\; 4\,\rho_{sw} \quad . \tag{8}$$

Adiabatic compression leads to the ratio

$$\frac{\rho_S}{\rho_B} \;=\; \left(\frac{T_S}{T_B}\right)^{1/(\gamma-1)} \;=\; \left(\frac{T_S}{T_B}\right)^{3/2} \;=\; \left(\frac{16}{15}\right)^{3/2} \;=\; 1.102 \quad . \tag{9}$$

From this we find

$$\rho_S \;\simeq\; \left(\frac{16}{15}\right)^{3/2} 4\,\rho_{sw} \;=\; 4.407\,\rho_{sw} \quad . \tag{10}$$

d. The flow angle is defined

$$\tan\theta_j \;=\; \frac{v_{t,j}}{v_{n,j}} \tag{11}$$

where v_t, the flow tangent to the shock, is the same on both sides. This leads to

$$\tan\theta_2 \;=\; \frac{v_t}{v_2} \;\simeq\; 4\,\frac{v_t}{v_1} \;=\; 4\tan\theta_1 \quad . \tag{12}$$

The deflection across the shock is

$$\Delta\theta \;=\; \theta_2 - \theta_1 \;\simeq\; 4\,\theta_1 - \theta_1 \;=\; 3\theta_1 \quad , \tag{13}$$

after using the small angle approximation to replace $\tan\theta_j \simeq \theta_j$.

e. The post-shock velocity vector is

$$\mathbf{v}_2 \;=\; v_{n,2}\hat{\mathbf{n}} + v_{t,2}\hat{\mathbf{t}} \;=\; v_{n,2}\,[\,\hat{\mathbf{n}} + \tan\theta_2\hat{\mathbf{t}}\,] \;=\; v_{n,2}\,[\,\hat{\mathbf{n}} + 4\tan\theta_1\hat{\mathbf{t}}\,] \tag{14}$$

where $\hat{\mathbf{n}}$ is the shock normal and $\hat{\mathbf{t}}$ a tangent vector perpendicular to it. The square magnitude is therefore

$$|\mathbf{v}_2|^2 \;=\; v_{n,2}^2\,[\,1 + 16\tan^2\theta_1\,] \quad . \tag{15}$$

Compared to the local sound speed this is

$$\frac{|\mathbf{v}_2|^2}{c_{s,2}^2} \;=\; \frac{v_{n,2}^2}{c_{s,2}^2}\,[\,1 + 16\tan^2\theta_1\,] \;=\; M_2^2\,[\,1 + 16\tan^2\theta_1\,] \;=\; \frac{1}{5}[\,1 + 16\tan^2\theta_1\,] \quad ,$$

using the hypersonic limit from eq. (1). Setting this to exceed one gives the requirement

$$\tan\theta_1 \;>\; \tfrac{1}{2} \quad , \quad \theta_1 > \tan^{-1}(1/2) \;=\; 26.5^\circ \quad . \tag{16}$$

f. The radial and poloidal velocities are

$$v_r \;=\; \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \;=\; 2\frac{A}{R_{mp}^2}\left(\frac{r^2}{R_{mp}^2} - \frac{R_{mp}^3}{r^3}\right)\cos\theta \quad , \tag{17}$$

$$v_\theta \;=\; -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \;=\; -\frac{A}{R_{mp}^2}\left(4\frac{r^2}{R_{mp}^2} + \frac{R_{mp}^3}{r^3}\right)\sin\theta \quad , \tag{18}$$

so evidently A is negative. The angle of the downstream flow relative to the shock normal, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, at $r = R_{bs}$, just inside the bow shock, is

$$\tan\theta_2 \;=\; -\frac{v_\theta}{v_r} \;=\; \frac{4(R_{bs}/R_{mp})^3 + (R_{mp}/R_{bs})^2}{2(R_{bs}/R_{mp})^3 - 2(R_{mp}/R_{bs})^2}\tan\theta \quad . \tag{19}$$

According to part d. this must be $\tan\theta_2 = 4\tan\theta_1$, and $\theta_1 = \theta$, to polar angle, since the flow is horizontal outside the shock. This leads to the relation

$$\frac{4(R_{bs}/R_{mp})^5 + 1}{2(R_{bs}/R_{mp})^5 - 2} \;=\; 4 \quad , \tag{20}$$

and therefore

$$R_{bs} \;=\; \left(\frac{9}{4}\right)^{1/5} R_{mp} \;=\; 1.18\,R_{mp} \quad . \tag{21}$$

This result is often cast in terms of the "stand-off" distance between the bow shock and magnetopause:

$$\Delta \;=\; R_{bs} - R_{mp} \;=\; 0.18\,R_{mp} \quad , \quad M_{sw} \to \infty \quad . \tag{22}$$

The simple model proposed above, a super-sonic flow encountering a spherical obstacle of radius R, is one for which there has been much study. Laboratory experiments with different Mach numbers and different fluids (i.e. differing γ) have led to empirical relations of the form

$$\frac{\Delta}{R} \;\simeq\; \alpha\,\frac{\rho_1}{\rho_2} \quad , \tag{23}$$

with values $\alpha \simeq 0.78$ (Seiff, *NASA Tech. Pub.* 24, [1962]). Numerical solutions of fully compressible hydrodynamics yield $\alpha \simeq 1.1$ (Spreiter, Summers & Alkse, *Planet. Space Sci.* **14** 223 [1966]), which has been subsequently used to predict the stand-off distance of the actual bow shock, at least for cases with high solar wind Mach number (see Farris & Russell, *JGR* **99**, 17681 [1994]). The simplified calculation we have performed here has found a lower value, $\alpha = 0.18 \times 4 = 0.704$, due mostly to the departure of actual post-shock flow from the incompressible solution we assumed. It is not, however, due to our assumption of a spherical magnetopause. As shown in part e., the sheath flow becomes supersonic beyond an angle of $\theta = 26^\circ$. Inside this small region there is little difference between a sphere and the actual magnetopuase. Differences outside that region can affect neither the subsonic solution nor the stand-off distance.