Problem set solutions: MHD dynamos

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1 Tensor algebra

a) Compute the double contraction $\varepsilon_{ijk}\varepsilon_{ijl}$.

Solution: Using $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ leads too:
$\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj} = 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}$.

b) Proof the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A}$$

Solution: Compute i^{th} component of expression:

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}A_l B_m) \\
&= \varepsilon_{kij}\varepsilon_{klm}\left(\frac{\partial A_l}{\partial x_j}B_m + A_l\frac{\partial B_m}{\partial x_j}\right) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\frac{\partial A_l}{\partial x_j}B_m + A_l\frac{\partial B_m}{\partial x_j}\right) \\
&= B_m\frac{\partial A_i}{\partial x_m} + A_i\frac{\partial B_m}{\partial x_m} - B_i\frac{\partial A_l}{\partial x_l} - A_l\frac{\partial B_i}{\partial x_l} \\
&= (\boldsymbol{B} \cdot \boldsymbol{\nabla})A_i + A_i\boldsymbol{\nabla} \cdot \boldsymbol{B} - B_i\boldsymbol{\nabla} \cdot \boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})B_i
\end{aligned}$$

c) Any anti-symmetric tensor, $a_{ij} = -a_{ji}$, has three independent components (i.e. the elements above the diagonal). It can therefore be expressed in terms of a 3-component vector using the Levi-Civita symbol, $a_{ij} = -\epsilon_{ijk}\gamma_k$. Derive an inverse expression given the vector γ_k explicityly in terms of a_{ij}.

Solution: Contract $a_{ij} = -\varepsilon_{ijk}\gamma_k$ with $-\frac{1}{2}\varepsilon_{lij}$:

$$-\tfrac{1}{2}\varepsilon_{lij}a_{ij} = \tfrac{1}{2}\varepsilon_{lij}\varepsilon_{ijk}\gamma_k = \tfrac{1}{2}\underbrace{\varepsilon_{ijl}\varepsilon_{ijk}}_{2\delta_{lk}}\gamma_k = \gamma_l$$

2 Second order correlation approximation

a) Start from the induction equation for $\boldsymbol{B}'$ (Volume I, Eq. 3.44):

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{v}} \times \boldsymbol{B}' - \eta\boldsymbol{\nabla} \times \boldsymbol{B}' + \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}\right) , \tag{1}$$

and assume $\overline{\boldsymbol{v}} = 0$, $|\boldsymbol{B}'| \ll |\overline{\boldsymbol{B}}|$ and neglect the contribution from magnetic resistivity. Formally integrate the equation to obtain a solution for $\boldsymbol{B}'$ and derive an expression for $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$. Assume that $\boldsymbol{v}'$ has a finite correlation time, τ_c, and simplify expressions by approximating time integrals with $\int_{-\infty}^{t} \overline{v_i'(t) v_k'(s)} \mathrm{d}s = \tau_c \overline{v_i'(t) v_k'(t)}$.

Solution:
The simplified induction equation reads:

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}}\right) \ ,$$

The formal solution for $\boldsymbol{B}'$ is:

$$\boldsymbol{B}'(t) = \int_{-\infty}^{t} \boldsymbol{\nabla} \times \left(\boldsymbol{v}'(s) \times \overline{\boldsymbol{B}(s)}\right) \mathrm{d}s \ .$$

The resulting emf reads:

$$\begin{aligned}\overline{\boldsymbol{\mathcal{E}}} &= \int_{-\infty}^{t} \overline{v'(t) \times \boldsymbol{\nabla} \times \left(\boldsymbol{v}'(s) \times \overline{\boldsymbol{B}}(s)\right)} \mathrm{d}s \approx \tau_c \overline{v' \times \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}}\right)} \\ &= \tau_c \overline{v' \times \left[(\overline{\boldsymbol{B}} \cdot \boldsymbol{\nabla})\boldsymbol{v}' - \overline{\boldsymbol{B}} \boldsymbol{\nabla} \cdot \boldsymbol{v}' - (\boldsymbol{v}' \cdot \boldsymbol{\nabla})\overline{\boldsymbol{B}}\right]}\end{aligned}$$

b) Express now all terms using the component notation summarized and show that the tensors a_{ij} and b_{ijk} in the expansion $\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$ are given by:

$$a_{ij} = \tau_c \overline{\left(\varepsilon_{ikl} v_k' \frac{\partial v_l'}{\partial x_j} - \varepsilon_{ikj} v_k' \frac{\partial v_m'}{\partial x_m}\right)} \tag{2}$$

$$b_{ijk} = \tau_c \varepsilon_{ijm} \overline{v_m' v_k'} \ . \tag{3}$$

Solution:

$$\begin{aligned}\overline{\mathcal{E}}_i &= \tau_c \overline{\left\{\varepsilon_{ikl} v_k' \left(\overline{B_j} \frac{\partial v_l'}{\partial x_j} - \overline{B_l} \frac{\partial v_m'}{\partial x_m}\right) - \varepsilon_{imj} v_m' v_k' \frac{\partial \overline{B}_j}{\partial x_k}\right\}} \\ &= \underbrace{\tau_c \overline{\left(\varepsilon_{ikl} v_k' \frac{\partial v_l'}{\partial x_j} - \varepsilon_{ikj} v_k' \frac{\partial v_m'}{\partial x_m}\right)}}_{a_{ij}} \overline{B_j} + \underbrace{\tau_c \varepsilon_{ijm} \overline{v_m' v_k'}}_{b_{ijk}} \frac{\partial \overline{B}_j}{\partial x_k}\end{aligned}$$

c) Decompose these tensors into the terms $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ defined through:

$$\begin{aligned} \alpha_{ij} &= \frac{1}{2}\left(a_{ij} + a_{ji}\right) \\ \gamma_i &= -\frac{1}{2}\varepsilon_{ijk} a_{jk} \\ \beta_{ij} &= \frac{1}{4}\left(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}\right) . \end{aligned}$$

Compute the trace α_{ii} and β_{ii}. To which physical quantities are they related?

Solution:

$$a_{ij} = \tau_c \overline{\left(\varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_j} - \varepsilon_{ikj} v'_k \frac{\partial v'_m}{\partial x_m} \right)}$$

Since the second term is antisymmetric in i and j, it does not contribute to α_{ij}. Thus we have:

$$\begin{aligned} \alpha_{ij} &= \frac{1}{2}\tau_c \overline{\left(\varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_j} + \varepsilon_{jkl} v'_k \frac{\partial v'_l}{\partial x_i} \right)} \\ \alpha_{ii} &= \tau_c \overline{\varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_i}} = -\tau_c \overline{v'_k \varepsilon_{kil} \frac{\partial v'_l}{\partial x_i}} = -\tau_c \overline{\boldsymbol{v}' \cdot \boldsymbol{\nabla} \times \boldsymbol{v}'} \end{aligned}$$

$$\begin{aligned} \gamma_n = -\frac{1}{2}\varepsilon_{nij} a_{ij} &= -\frac{1}{2}\tau_c \overline{\left(\varepsilon_{nij}\varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_j} - \varepsilon_{nij}\varepsilon_{ikj} v'_k \frac{\partial v'_m}{\partial x_m} \right)} \\ &= -\frac{1}{2}\tau_c \overline{\left(\underbrace{\varepsilon_{ijn}\varepsilon_{ikl}}_{\delta_{jk}\delta_{nl} - \delta_{jl}\delta_{nk}} v'_k \frac{\partial v'_l}{\partial x_j} + \underbrace{\varepsilon_{nij}\varepsilon_{kij}}_{2\delta_{kn}} v'_k \frac{\partial v'_m}{\partial x_m} \right)} \\ &= -\frac{1}{2}\tau_c \overline{\left(v'_k \frac{\partial v'_n}{\partial x_k} - v'_n \frac{\partial v'_j}{\partial x_j} + 2 v'_n \frac{\partial v'_j}{\partial x_j} \right)} \\ &= -\frac{1}{2}\tau_c \frac{\partial}{\partial x_m} \overline{v'_n v'_m} \end{aligned}$$

With $b_{ijk} = \tau_c \varepsilon_{ijm} \overline{v'_m v'_k}$ we get:

$$\begin{aligned} \beta_{ij} &= \frac{1}{4}\left(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}\right) = \frac{1}{4}\tau_c \left(\varepsilon_{ikl}\varepsilon_{jkm} + \varepsilon_{jkl}\varepsilon_{ikm}\right) \overline{v'_m v'_l} \\ &= \frac{1}{2}\tau_c \varepsilon_{ikl}\varepsilon_{jkm} \overline{v'_m v'_l} = \frac{1}{2}\tau_c \left(\delta_{ij}\delta_{lm} - \delta_{im}\delta_{jl}\right) \overline{v'_m v'_l} \\ &= \frac{1}{2}\tau_c \left(\delta_{ij} \overline{v'^2} - \overline{v'_i v'_j} \right) \\ \beta_{ii} &= \tau_c \overline{v'^2} \end{aligned}$$

α_{ii} is proportional to the negative kinetic helicity of the flow, β_{ii} is proportional to the turbulent rms velocity squared. γ can be expressed as the divergence of the velocity correlation tensor.

d) Make now the additional assumption of isotropy, which implies that α_{ij}, β_{ij}, as well as the correlation tensor $\overline{v'_i v'_j}$ are diagonal, i.e. $\alpha_{ij} = \alpha\delta_{ij}$. Compute the scalar α-effect and the turbulent diffusivity η_t.

How is $\boldsymbol{\gamma}$ related to η_t? Discuss under which conditions these effects exist.

Solution:
Isotropy implies:

$$\begin{aligned}
\alpha &= \frac{1}{3}\alpha_{ii} = -\frac{1}{3}\tau_c \overline{\boldsymbol{v'} \cdot \boldsymbol{\nabla} \times \boldsymbol{v'}} \\
\eta_t &= \frac{1}{3}\beta_{ii} = \frac{1}{3}\tau_c \overline{v'^2} \\
\gamma_i &= -\frac{1}{2}\tau_c \frac{\partial}{\partial x_m}\overline{v'_i v'_m} = -\frac{1}{2}\tau_c \frac{\partial}{\partial x_m}\left(\frac{1}{3}\overline{v'^2}\delta_{im}\right) \\
&= -\frac{1}{6}\tau_c \frac{\partial}{\partial x_i} v'^2 = -\frac{1}{2}\frac{\partial}{\partial x_i}\eta_t
\end{aligned}$$

Note that the last step is only valid if τ_c does not vary spatially. Although this effect is very often expressed as gradient of η_t, this is not the case for highly stratified convection such as the solar convection zone. Since v'^2 is increasing monotonically from the base of the CZ toward the photosphere, the resulting γ describes a downward transport throughout the entire CZ "turbulent pumping".

η_t is present under minimal assumptions (e.g. isotropy, homogeneity) since it is simply related to the turbulence intensity. γ requires in addition inhomogeneity (e.g. stratification). For α reflectional symmetry needs to be broken, e.g. through a combination of stratification and rotation.

3 Biermann battery

From the equation of motion for the drift velocity $\boldsymbol{v}_d$ of electrons

$$n_e m_e \left(\frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}}\right) = n_e q_e(\boldsymbol{E} + \boldsymbol{v}_d \times \boldsymbol{B}) - \boldsymbol{\nabla} p_e$$

with:
τ_{ei}: collision time between electrons and ions
n_e: electron density
q_e: electron charge
m_e: electron mass
p_e: electron pressure
$\boldsymbol{E}$: electric field in local frame of rest of fluid,
we can derive an expression for the electric current $\boldsymbol{j} = n_e\, q_e\, \boldsymbol{v}_d$ (generalized Ohm's law):

$$\frac{\partial \boldsymbol{j}}{\partial t} + \frac{\boldsymbol{j}}{\tau_{ei}} = \frac{n_e q_e^2}{m_e}\boldsymbol{E} + \frac{q_e}{m_e}\boldsymbol{j} \times \boldsymbol{B} - \frac{q_e}{m_e}\boldsymbol{\nabla} p_e$$

For simplicity we neglect in the following the time derivative of $\boldsymbol{j}$ (no plasma oscillations) and the Hall term (second term on the right hand side). In addition we express the equation in the laboratory frame, by substituting $\boldsymbol{E}$ by $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$. Solving for $\boldsymbol{E}$ gives:

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \frac{1}{\sigma}\boldsymbol{j} - \frac{1}{\varrho_e}\boldsymbol{\nabla} p_e\ .$$

Here, $\sigma = \tau_{ei} n_e q_e^2 / m_e$ denotes the electric conductivity, and $\varrho_e = n_e\, q_e$ the electron charge density. Using Maxwell's second law yields the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B} - \eta\, \boldsymbol{\nabla} \times \boldsymbol{B}) + \frac{1}{\varrho_e^2}\boldsymbol{\nabla}\varrho_e \times \boldsymbol{\nabla} p_e\ .$$

The magnetic field independent source term $\boldsymbol{\nabla}\varrho_e \times \boldsymbol{\nabla} p_e/\varrho_e^2$ is formally identical to the baroclinic term in the vorticity equation. Starting from the momentum equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\frac{v^2}{2} - \boldsymbol{v} \times \boldsymbol{\nabla} \times \boldsymbol{v} = -\frac{1}{\varrho}\boldsymbol{\nabla} p$$

the equation for $\omega = \boldsymbol{\nabla} \times \boldsymbol{v}$ is given by

$$\frac{\partial \omega}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \omega) + \frac{1}{\varrho^2}\boldsymbol{\nabla}\varrho \times \boldsymbol{\nabla} p \, .$$

The term $\frac{1}{\varrho^2}\boldsymbol{\nabla}\varrho \times \boldsymbol{\nabla} p$ is often referred to as "baroclinic vector". This term vanishes for a barotrope fluid in which $p = p(\varrho)$. In the Earths's atmosphere baroclinic conditions are found mostly in mid-latitudes, where front systems often lead to rapid temperature changes that are not aligned with constant pressure surfaces.

Going back to the induction equation, a contribution from $\frac{1}{\varrho_e^2}\boldsymbol{\nabla}\varrho_e \times \boldsymbol{\nabla} p_e$ can arise in the universe when bright point sources (quasars in the early universe, hot young stars in star formation regions) drive ionization fronts through an inhomogeneous plasma (the background density fluctuations are independent from the orientation of the ionization fronts). It has been estimated by Subramanian et al. 1994 (MNRAS 271, 15) that this process can produce magnetic field in the intergalactic medium of the order of $3 \cdot 10^{-23}$ G, which would lead to a galactic seed field of the order of $3 \cdot 10^{-20}$ G after a density fluctuation collapsed and formed a galaxy (amplification by a factor of about 10^3). Such a weak seed field would be sufficient to explain the observed magnetic field of galaxies $\sim 10^{-6}$ G, assuming that a dynamo exponentiated the field over 30 times. The latter would require a growth rate of ~ 3 Gyr^{-1}, which is within the realm of estimates for galactic dynamos (see for example the extensive review by Brandenburg & Subramanian 2005, Physics Reports, Volume 417, Issue 1-4, p. 1-209, astro-ph/0405052).