1. **Turbulent Heating**. Matthaeus & Goldstein (JGR, **87**, 6011, 1982) analyzed an interval of slow-solar-wind data taken by instruments onboard the *Voyager* spacecraft in 1977. In this data interval, the mean flow speed was 352 km/s, the mean proton number density was 12.6 $\text{cm}^{-3}$, the rms amplitude of the velocity fluctuations was $\delta v = 36.7$ km/s, and the velocity correlation length was $L_c = 2.83 \times 10^6$ km. In this problem, you can think of $L_c$ as the largest-eddy size and $\delta v$ as the approximate velocity of eddies of size $L_c$. (Here you are making use of the fact that most of the velocity fluctuation energy is in the largest eddies.)

   (a) Use the Kolmogorov estimate for the turbulence cascade power to estimate the turbulent heating rate per unit mass (in $\text{g}^{-1}$ erg $\text{s}^{-1}$) for this interval of data. Note that this formula provides only a very rough estimate of the actual heating rate. Compare your answer to the heating rate that is needed to explain the non-adiabatic temperature profile of the solar wind. (See the earlier problem from Dr. Kasper.) Based on this comparison, is it plausible that turbulent heating could be the dominant heating mechanism in the solar wind near 1 AU?

   (b) Suppose you were to look at the time series of velocity measurements for this data set and calculate the temporal auto-correlation function

$$C(\tau) = \frac{\langle \vec{v}(t) \cdot \vec{v}(t+\tau) \rangle}{\langle \vec{v}(t) \cdot \vec{v}(t) \rangle}.$$

   The angle brackets $\langle \dots \rangle$ here indicate an average over the time $t$. Note that $C = 1$ for $\tau = 0$ and $C \to 0$ as $\tau \to \infty$. (The latter is true because velocity fluctuations measured at sufficiently widely separated times are uncorrelated.) One definition of the "correlation time" $\tau_c$ is that $\tau_c$ satisfies the relation $C(\tau_c) = e^{-1} = 0.368$. What is the correlation time for the velocity data considered by Matthaeus & Goldstein (1982)? **Hint:** this question relates to Taylor's frozen-in-flow hypothesis. If you were to take a snapshot of the solar wind and record the value of $\vec{v}$ along the line connecting your spacecraft to the Sun, you could compute a spatial autocorrelation function $C_{\text{spatial}}$ analogous to the above temporal autocorrelation function, but with $t \to r$ and $\tau \to \delta r$. The spatial correlation length (i.e., $L_c$) would then be that value of $\delta r$ for which $C_{\text{spatial}}(\delta r) = e^{-1}$.

2. **Critical Balance.** For many years, the heliospheric physics community favored the hypothesis that the dominant source of proton heating in the solar corona and solar wind is resonant cyclotron heating by high-frequency waves. The idea here is that when a wave has an angular frequency $\omega$ equal (or nearly equal) to the proton cyclotron frequency,

$$\Omega_p = \frac{qB}{m_p c}, \tag{1}$$

   a proton's gyromotion is in resonance with the electric-field fluctuation associated with the wave, leading to a strong interaction between the proton and the wave. In this case, protons are strongly heated by the wave, and the wave is strongly damped by the protons. This hypothesis of cyclotron heating has been called into question by evidence suggesting that

Alfvén-wave turbulence is responsible for much of the proton heating, and by the discovery that the Alfvén-wave energy cascade is anisotropic, with $\lambda_\perp \ll \lambda_\parallel$ at the small lengthscales at which the Alfvén-wave turbulence dissipates. In this problem, you will explore this issue by drawing upon the idea of "critical balance" in Alfvén-wave turbulence.

(a) Let us consider again the interval of slow-solar-wind data described above from Matthaeus & Goldstein (1982). Imagine that the turbulence is strong and isotropic at the "outer scale" $L_\mathrm{c} = 2.83 \times 10^6$ km with $\chi = 1$. In other words, assume that at perpendicular scale $\lambda_\perp = L_\mathrm{c}$, the Alfvén wave packets are isotropic with $\lambda_\perp = \lambda_\parallel$ and $\delta v_{\lambda_\perp} = v_\mathrm{A}$. Now, using the results in the lecture notes, derive an expression for the value of $\lambda_\parallel$ as a function of $\lambda_\perp$ for values of $\lambda_\perp < L_\mathrm{c}$.

(b) In magnetized plasmas, protons move along helical paths centered on magnetic field lines. The distance between one of these helical paths and the central magnetic field line is the proton gyroradius, $\rho_\mathrm{p}$, which is given by

$$\rho_\mathrm{p} = \frac{v_\perp}{\Omega_\mathrm{p}},$$

where $v_\perp$ is the speed of the proton in the plane perpendicular to the magnetic field. In this problem, you can take $v_\perp$ to be the proton thermal speed $\sqrt{k_\mathrm{B} T_\mathrm{p}/m_\mathrm{p}}$. Evaluate $\Omega_\mathrm{p}$ and $\rho_\mathrm{p}$ assuming that $B = 6 \times 10^{-5}$ G and $T = 10^5$ K. ($k_\mathrm{B} = 1.38 \times 10^{-16}$ erg K$^{-1}$; $m_\mathrm{p} = 1.67 \times 10^{-24}$ g; $c = 3 \times 10^{10}$ cm/s; $q = 4.80 \times 10^{-10}$ statcoul)

(c) Combining your results from the previous two questions, evaluate $\lambda_\parallel$ when $\lambda_\perp = \rho_\mathrm{p}$.

(d) Evaluate $v_\mathrm{A} = B/\sqrt{4\pi\rho}$ assuming that $B = 6 \times 10^{-5}$ G and that protons make up all of the mass density, with a mean proton number density of 12.6 cm$^{-3}$.

(e) The frequency of an Alfvén wave is given by $\omega = k_\parallel v_\mathrm{A}$. For strongly turbulent Alfvén wave packets, you can estimate a characteristic wave frequency as $\omega_\mathrm{eff} \sim v_\mathrm{A}/\lambda_\parallel$. For critically balanced Alfven wave turbulence with $\lambda_\perp \in (\rho_\mathrm{p}, L_\mathrm{c})$, at which value of $\lambda_\perp$ will $\omega_\mathrm{eff}$ have the highest value? Using your results from the above questions, compute $\omega_\mathrm{eff}$ for wavepackets with $\lambda_\perp = \rho_\mathrm{p}$ and compare this value with $\Omega_\mathrm{p}$.

(f) Based on this comparison, do you expect strong Alfvén wave turbulence with $\lambda_\perp \in (\rho_\mathrm{p}, L_\mathrm{c})$ to lead to efficient cyclotron heating of protons in the solar wind? Note: it's possible that there could be some cyclotron heating by fluctuations with $\lambda_\perp \ll \rho_\mathrm{p}$, but one argument against this possibility is that the protons tend to average over such fluctuations during their gyromotion, so that they don't "feel" the fields in those fluctuations very strongly. When $\lambda_\perp \lesssim \rho_\mathrm{p}$, other dissipation/heating mechanisms (Landau damping, transit-time damping, and stochastic heating) become important, and these may be more important than proton cyclotron heating for dissipating the turbulence. It should be noted that at $\lambda_\perp \lesssim \rho_\mathrm{p}$, the nature of the fluctuations change — in particular, at these small "kinetic scales," there is no Alfvén wave. There are, however, kinetic Alfvén waves, whistler waves, Bernstein modes, and a number of other wave types. The nature and dissipation of turbulence at $\lambda_\perp \lesssim \rho_\mathrm{p}$ is an area of active research, with many interesting, open questions.