1 Estimating coronal temperatures using Parker model

In the lecture we saw that the speed of the solar wind near Earth at 1 AU ranges from about 275 km/s to 800 km/s. Estimate the corresponding range of coronal temperatures T_o that produce these speeds using Parker's model. *Hint: The approximation provided in the lecture for the speed at large distances from the Sun is valid at 1 AU*

Start with the approximate expression for the speed of the solar wind far from the Sun, $v \simeq 2C_S\sqrt{\ln r/r_c}$. Recall that the sound speed $C_S^2 = k_B T_o/M$, where $M \simeq 0.5m_p$ is the mean molecular mass of a proton-electron plasma, and $r_c = GM_\odot M/2k_B T_o$. Combining these equations, we have an expression for the solar wind speed at 1 AU:

$$v(r, T_o) = 2\sqrt{\frac{k_B T_o}{0.5m_p} \ln \frac{2k_B T_o}{GM_\odot 0.5m_p} r}$$

Temperature [MK]	Speed at 1 AU [km s^{-1}]
0.4	**267**
0.6	408
0.8	514
1.0	606
1.2	689
1.4	766
1.5	**803**

And the corresponding range of temperatures is $0.4 - 1.5$ MK.

2 Where does the solar wind begin?

2.1 Estimating sonic critical point

Is there a clearly defined interface between the solar corona and the solar wind? One useful definition is that the solar wind begins when it is flowing away from the Sun so quickly that waves generated within the wind are not able to communicate back to the Sun. Using the range of temperatures calculated in Q1 and the isothermal Parker model for supersonic wind acceleration, calculate the corresponding critical points r_c from the Sun where the plasma undergoes a transition from subsonic to supersonic flow. Express your results in units of solar radii.

Using $r_c = GM_\odot M/2k_B T_o$ and the temperatures of 0.4 and 1.5 MK calculated in Q1, we calculate the locations of the sonic transition point:

Temperature [MK]	Speed at 1 AU [km s^{-1}]	Sonic Point [R_s]
0.4	267	14.4
0.6	408	9.6
0.8	514	7.2
1.0	606	5.8
1.2	689	4.8
1.4	766	4.1
1.5	803	3.9

Note that the isothermal Parker model contains many simplifications, and therefore these estimates of the sonic point are rough approximations.

2.2 Alfvén critical point

Since the corona and solar wind are magnetized, information may also propagate via Alfvén waves. A more appropriate critical point therefore may be when the bulk speed of the wind exceeds the Alfvén speed C_A. If $\beta < 1$ in the solar corona, is the Alfvén point closer or further from the Sun than the sonic point? Which critical point sets the location where the Sun no longer controls the solar wind?

Start by recalling that β is the ratio of particle pressure to magnetic pressure, $\beta = 2\mu_o n k_B T/B^2$. For an ideal gas, $C_s^2 = \gamma p/\rho$. Since $p = nk_B T$ and $\rho = nm_p$, $C_s^2 = \gamma k_B T/m_p$. The Alfvén speed is given by $C_A^2 = B^2/\mu_o\rho = B^2/\mu_o m_p n_p$. We see then that $\beta = \frac{2}{\gamma} C_s^2/C_a^2$, so $C_a \simeq C_s\beta^{-1/2}$if $\gamma \sim \frac{5}{3}$. If $10^{-2} < \beta < 10^{-1}$ in the upper corona, then $C_A$is about 3-10 times further from the Sun than the sonic point. Estimates place the Alfvén critical point between $10 - 20R_\odot$. Solar Probe Plus has a minimum perihelion of $8.5R_\odot$ from the surface of the Sun so it will reach below the Alfvén critical point and enter the region of the inner heliosphere controlled by the Sun's magnetic field.

3 Estimate of heating rate

In the lecture we made use of the conservation and momentum equations to derive the isothermal Parker solar wind solution. Now we will briefly examine the energy equation to get a sense of the additional heating occurring in the solar wind as it expands from the Sun. We begin with the MHD equation for internal energy

$$\frac{\rho^\gamma}{\gamma - 1}\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = -L\,, \quad (1)$$

where γ is the polytropic index ($\gamma = \frac{5}{3}$ for an ideal gas) and L is the heal loss function (all sources and sinks of heat). Written generally, $L = \nabla \cdot \vec{q} - Q$, where $\vec{q}$ represents the conduction of heat through the plasma and Q represents a local heating rate due to e.g. the dissipation of turbulence. Note that if there is no additional heating, $L = 0$ and we recover the usual adiabatic invariant $\frac{p}{\rho^\gamma} = const.$

If we assume the solar wind is in steady state, spherically symmetric, and expanding at a constant speed U, and that heat conduction plays a small role at 1 AU (i.e. $\vec{\nabla} \cdot \vec{q} = 0$), then the heat equation takes the form

$$\frac{dT}{dr} = \frac{2}{r}(1-\gamma)T + \frac{\gamma - 1}{U\rho}\frac{M}{k_B}Q \quad (2)$$

3.1 Radial profile from adiabatic expansion

Calculate the expected radial dependence of T on r at 1 AU assuming there is no local heating (Q=0). Take (2) above and show that that the temperature will have a power law dependence on distance, $T \propto r^{-\alpha}$. Calculate the value of α for an ideal gas with $\gamma = 5/3$. Add your prediction to the table below, and compare adiabatic expansion with the predictions from the three models we considered in the lecture, and the observed electron and proton temperature gradients.

Model	Temperature power law index at 1 AU
Chapman static atmosphere	0.29
Chamberlin exospheric evaporation	0.50
Parker isothermal hydrodynamic	0.00
Adiabatic expansion	
Observed proton temperature (Helios and Ulysses)	**0.80**
Observed electron temperature (Helios and Ulysses)	**0.43**

Starting with (2), but with $Q = 0$, $\frac{dT}{dr} = \frac{2}{r}(1-\gamma)T$, or $\frac{dT}{T} = 2(1-\gamma)\frac{dr}{r}$. Solutions to this equation will take the form $T \propto r^{2(1-\gamma)}$. For $\gamma = \frac{5}{3}$, $\alpha = 1.33$. This is a much faster predicted drop in temperature than observed, both for protons and electrons. Additional heating must be taking place in the solar wind, even at 1 AU, for this gradient to be produced, although less than that resulting from the heat conduction in the static and evaporation models. By definition the isothermal Parker model does not have anything to say about temperature gradients.

3.2 Observed heating rate

Now we will use the full version of equation (2) to estimate the level of additional heating occurring in the solar wind based on observed temperature gradients.

We will use empirical values for the observed solar wind proton and electron temperature gradients in high speed wind near 1 AU, as measured with the Helios and Ulysses spacecraft and reported in Cranmer et al. (2009). For the selected observations, the mean speed was $U = 705$ km s^{-1}, the number density at 1 AU was $n = 2.5$ cm^{-3} and:

$$T_p = 1.27 \times 10^5 \left(\frac{r}{1AU}\right)^{-0.80} K \quad (3)$$

$$T_e = 0.73 \times 10^5 \left(\frac{r}{1AU}\right)^{-0.43} K \quad (4)$$

Some additional flaws in our treatment of the solar wind to date are clearly apparent: protons and electrons have different temperatures, and different temperature profiles. Complete theories need to treat the interactions between species with different properties, but we will neglect this for today. Use the energy equation, the assumption that the solar wind is an ideal gas with $\gamma = 5/3$, and the reported temperature gradients, estimate the heating rate using the electron results and proton results separately. Express the results in $J\ m^{-3} s^{-1}$ and $erg\ g^{-1} s^{-1}$ for easy comparison with the next problem set.

Begin with the energy equation

$$\frac{dT}{dr} = \frac{2}{r}(1-\gamma)T + \frac{\gamma - 1}{U\rho}\frac{M}{k_B}Q$$

We solve for Q,

$$Q = \frac{U\rho}{\gamma - 1}\frac{k_B}{M}\left[\frac{dT}{dr} - \frac{2}{r}(1-\gamma)T\right]$$

We have T_p and T_e above and can take their derivatives easily enough (but make sure to account for the fact that r is in AU in (3) and (4) and convert into the correct physical units. $\rho \simeq m_p n$ and $M = 0.5 m_p$ so the proton masses cancel out.

Using	$Q\ [J\ m^{-3} s^{-1}]$	$Q\ [erg\ g^{-1} s^{-1}]$
Proton temperature	6.6×10^{-17}	1.6×10^{8}
Electron temperature gradient	6.4×10^{-17}	1.5×10^{8}

To put this in perspective, let's consider how quickly a glass of water would change temperature if it were being heated at this rate. The specific heat of water is $c_p = 4.18 \times 10^7\ erg\ g^{-1}\ C^{-1}$. The time rate of change in temperature is $\frac{dT}{dt} = \frac{1}{c_p} Q \simeq 3.8°C\ s^{-1}$. So the dissipation mechanism acting in the solar wind could boil a cup of water for you in less than 15 seconds, if of course the same physics held in your tea cup. Not too shabby!