Solutions to Problems for Magnetic Energy Conversion Processes

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Problem 1.

(a) Since ρ is constant and uniform, the continuity equation reduces to

$$\nabla\bullet\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 .$$

which is satisfied when the expression for u_x and u_y are substituted into the equation.

(b) Since **B** does not vary in time, Faraday's equation reduces to

$$\nabla\times\mathbf{E} = \mathbf{0} .$$

Thus,

$$\frac{dE_z}{dx} = 0 \quad \text{and} \quad \frac{dE_z}{dy} = 0$$

These conditions are both satisfied if E_z = constant = $-E_0$.

(c) From Ampère's Law, the current density is

$$\mathbf{j} = -\frac{1}{\mu_0}\frac{\partial B_x}{\partial y}\,\hat{\mathbf{z}} ,$$

Substitution into Ohm's Law along with the expressions for **u** and **B**, yields

$$\frac{\partial B_x}{\partial y} + \frac{k\mu_0}{\eta_e}\,yB_x = \frac{E_0\mu_0}{\eta_e} ,$$

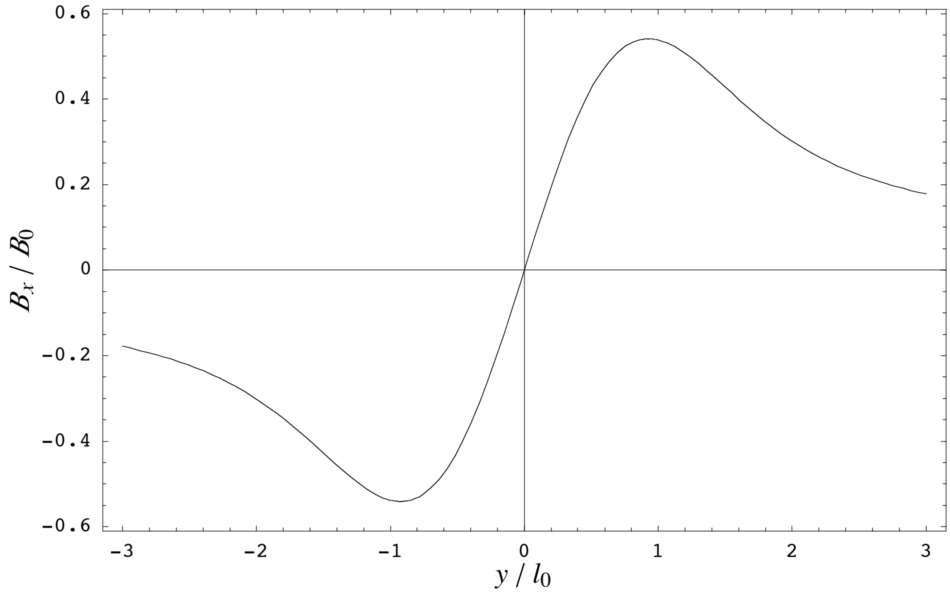
which is a first-order, linear , ordinary differential equation (ODE) with the solution

$$B_x = \frac{E_0\mu_0 l_0}{\eta_e}\,e^{-(y/l_0)^2}\int_0^{y/l_0} e^{t^2}\,dt = \frac{E_0\mu_0 l_0}{\eta_e}\,\mathrm{daw}(y/l_0) ,$$

where $l_0 = \sqrt{2\eta_e / k\mu_0}$ and daw(y/l_0) is the Dawson Integral function. Other notations used for this function are

$$\mathrm{daw}(x) = D_+(x) = \frac{1}{2}\sqrt{\pi}e^{-x^2}\mathrm{erfi}(x) = -\frac{1}{2}i\sqrt{\pi}e^{-x^2}\;\mathrm{erf}(ix) ,$$

(d)



The constant l_0 is the diffusive scale length, that is, the location where the outward diffusion of the magnetic field roughly equals the inward motion of the plasma. The constant B_0 is approximately 1.85 times the maximum value of B_x. This maximum value occurs at $y/l_0 = 0.924$ which is of order unity. Thus B_0 is about twice the maximum value of the magnetic field, and this maximum occurs close to the location where the outward diffusion of the field is balanced by the compression of the field due to the plasma inflow.

(e) The two components of the momentum equation are

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\partial y} - \frac{B_x}{\mu_0} \frac{\partial B_x}{\partial y} \ ,$$

and

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} \ .$$

Substitution of the expressions for u_x and u_y, leads to

$$-k^2 x = \frac{\partial p}{\partial x} \ , \quad \text{and} \quad -k^2 y = \frac{\partial}{\partial y}\left(p + \frac{B_x^2}{2\mu_0} \right)$$

which are both satisfied when

$$p = p_0 - \frac{1}{2} k^2 (x^2 + y^2) - \frac{1}{2} \frac{B_x^2}{2\mu_0}$$

where B_x is given by the solution to part (c). At large distance p tends to minus infinity primarily because the kinetic energy increases as distance squared. Thus, the solution is only valid for finite distances from the stagnation point at $x = 0$, $y = 0$.

Problem 2.

(a) Dropping the radiation term yields a partial differential equation (PDE) that can be solved using the method of separation of variables:

$$m_i n c_p \frac{\partial T}{\partial t} = \lambda \frac{\partial}{\partial s}\left(T^{5/2} \frac{\partial T}{\partial s} \right)$$

For an ideal gas at constant pressure $n = n_0\, T_0/T$, so the PDE for T becomes

$$\frac{m_i n_0 T_0 c_p}{\lambda} \frac{\partial T}{\partial t} = T \frac{\partial}{\partial s}\left(T^{5/2} \frac{\partial T}{\partial s} \right)$$

Setting $T(s, t) = f(t)\, g(s)$ yields:

$$\frac{c_p m_i n_0 T_0}{\lambda} f^{-9/2} \frac{df}{dt} = \frac{d}{ds}\left(g^{5/2} \frac{dg}{ds} \right) = K$$

where K is the separation constant. Solving the ODE for f gives

$$f = \left(C_1 - \frac{7K\lambda}{2c_p m_i n_0 T_0}\, t \right)^{-2/7}$$

where C_1 is a constant of integration. Solving the ODE for g gives

$$g = \left(\frac{7}{4} K s^2 + C_2\, s + C_3 \right)^{2/7}$$

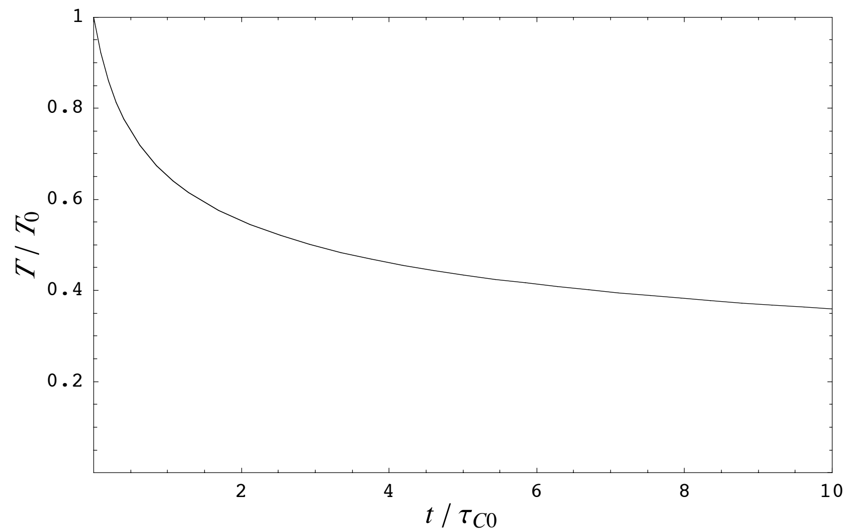
where C_2 and C_3 are also constants of integration. The solution for T is therefore

$$T(s,t) = \left(C_1 - \frac{7K\lambda}{2c_p m_i n_0 T_0}\, t \right)^{-2/7} \left(\frac{7}{4} K s^2 + C_2\, s + C_3 \right)^{2/7}.$$

To evaluate the three integration constants, C1, C2, and C3 we use the three conditions:

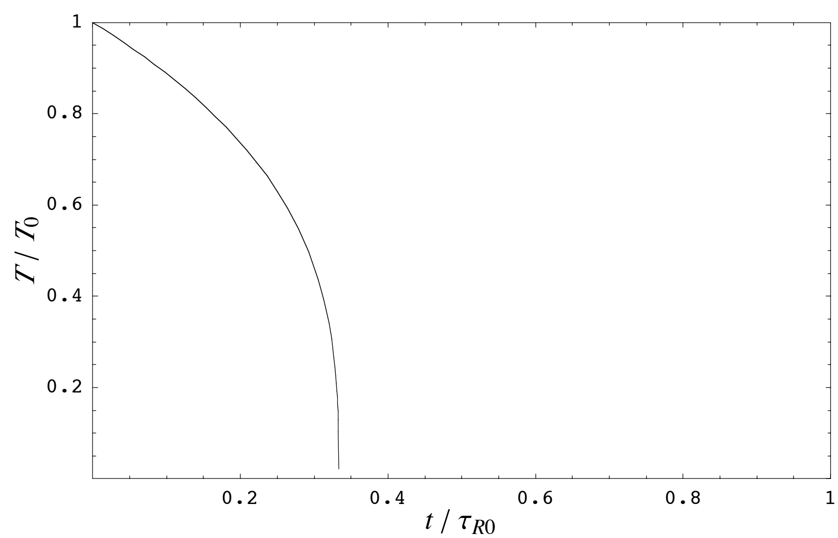
$$T(0,\ t)\ =\ 0; \qquad \left.\frac{\partial T(s,\ t)}{\partial s}\right|_{s=L}\ =\ 0; \qquad T(L,\ 0)\ =\ T_0\,.$$

The first condition gives $C_3 = 0$, the second condition gives $C_2 = -7KL/2$, and the third condition gives $C_1 = -\,(7/4)\ KL^2\ T_0^{-2/7}$. Substitution of the values into the above expression yields the solution given in part (a) of the problem. The value of K is not needed, because it cancels out of the equations. The cancellation occurs because of the problem's relatively simple initial and boundary conditions.



(b) In the absence of thermal conduction

$$m_i c_p \frac{\partial T}{\partial t} = -\,n\chi T^{\alpha} = -\,n_0 T_0\,\chi T^{\alpha-1}\ .$$



Setting $\alpha\ =\ -1$, and integrating yields the solution:

$$T = T_0 \left(1 - 3\frac{t}{\tau_{R0}} \right)^{1/3} ; \qquad \tau_{R0} = \frac{c_p m_i T_0^2}{n_0 \chi} .$$

The key difference between conductive and radiative cooling is that conductive cooling becomes slower as the temperature decreases while radiative cooling becomes faster. Given enough time, radiative cooling will eventually dominate, no matter how slow it is initially.

(c) The initial (i.e. linear) conductive and radiative cooling times for these values are:

$$\tau_{C0} = 1.23 \text{ s} \quad \text{and} \quad \tau_{R0} = 1.55 \times 10^5 \text{ s} \quad \text{with} \quad \tau_{R0}/\tau_{C0} = 1.27 \times 10^5.$$

Initially the radiative cooling is more than a hundred thousand times slower than the conductive cooling.

(d) At the loop top

$$T = T_0 \left(1 + \frac{7}{2}\frac{t}{\tau_{C0}} \right)^{-2/7}$$

so the local, nonlinear cooling time there is

$$\tau_C = \tau_{C0} \left(1 + \frac{7}{2} \frac{t}{\tau_{C0}} \right) = \tau_{C0} \left(\frac{T}{T_0} \right)^{-7/2}$$

The local, nonlinear radiative time is

$$\tau_R = \tau_{R0} \left(1 - 3 \frac{t}{R_{C0}} \right) = \tau_{R0} \left(\frac{T}{T_0} \right)^{3}$$

The two cooling times are equal when

$$\frac{T_{sw}}{T_0} = \left(\frac{\tau_{R0}}{\tau_{C0}} \right)^{-2/13}$$

where T_{sw} is the temperature where the switch over occurs. For pure conductive cooling, the switch over time, t_{sw}, is

$$\frac{t_{sw}}{\tau_{C0}} = \frac{2}{7} \left[\left(\frac{\tau_{R0}}{\tau_{C0}} \right)^{7/13} - 1 \right]$$

The plasma has to cool by about a factor of 6 before radiative cooling becomes important, i.e.

$$T_0/T_{sw} = 6.10\ ,$$

corresponding to a temperature of $T_{sw} = 4.92 \times 10^6$ K. The time of the switch over is

$$t_{sw} = 196 \text{ s}\ .$$

Thus, when a heated loop disconnects from the reconnection site, it only takes about three minutes before radiative cooling starts to dominate.

<u>Problem 3</u>.

The x and y components of Faraday' equation are:

$$\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} \quad \text{and} \quad \frac{\partial B_y}{\partial t} = -\frac{\partial E_z}{\partial x}\ .$$

Using Leibniz's rule (also known as the fundamental theorem of calculus), we can write the x component of Faraday's equation as

$$E_z(0,y_0) - E_z(0,0) = \frac{\partial}{\partial t}\int_0^{y_0} B_x(0,y)\ dy - B_x(0,y_0)\dot{y}_0$$

where y_0 is the location of the x-line. Because the field is line-tied at $y = 0$, $E_z(x, 0) = 0$. Also by definition $B_x\ (0, y_0) = 0$ and $E_z(0, y_0) = E_0$. Consequently,

$$E_0 = \frac{\partial}{\partial t}\int_0^{y_0} B_x(0,y)\ dy$$

Similarly, along the x-axis, we have

$$E_z(0,0) - E_z(x_0,0) = 0 = \frac{\partial}{\partial t}\int_0^{x_0} B_y(x,0)\ dx - B_y(x_0,0)\ \dot{x}_0$$

Since the magnitudes of the magnetic flux between $y = 0$ and y_0 and between $x = 0$ and x_0 are equal, we obtain

$$\dot{x}_0 = E_0\ /\ B_y(x_0,0)$$

Substitution of the given function into the above expression leads to:

$$\dot{x}_0 = \frac{dx_0}{dt} = \frac{E_0}{B_0\ a^3}\ \frac{(x_0^2 + a^2)^2}{x_0}$$

which upon integration yields:

$$-(1 + x_0^2/a^2) = t/\tau_A + \text{constant}$$

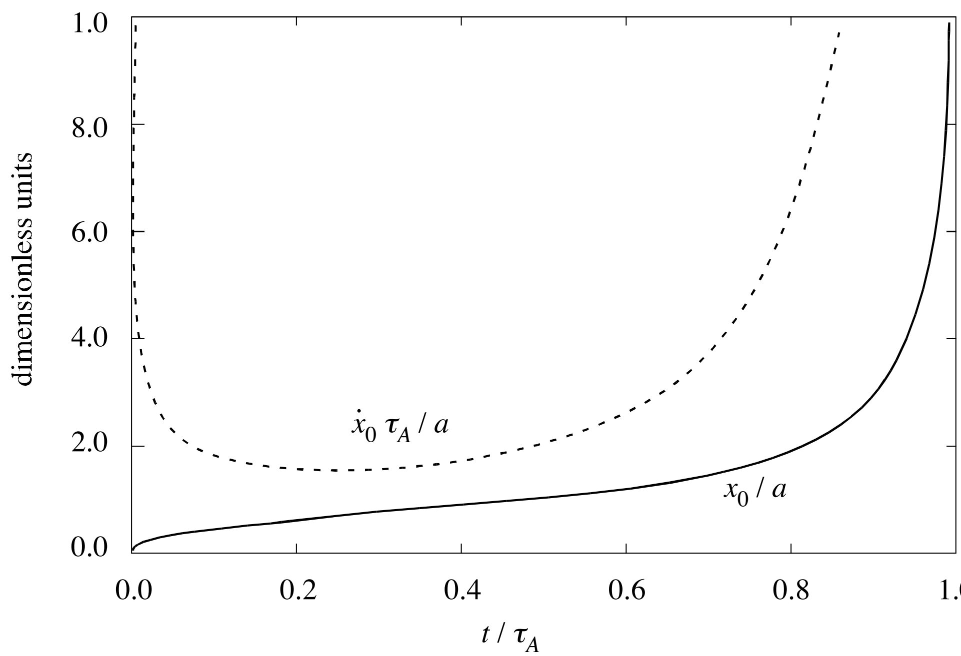
where $\tau_A = B_0\, a/(2E_0)$. Since it is assumed that $x_0 = 0$ at $t = 0$, the value of the constant is -1. So the solution for x_0 is

$$x_0 = a\sqrt{t/(\tau_A - t)}$$

and the solution for $\dot{x}_0$ is

$$\dot{x}_0 = (a/2\tau_A)\,(\tau_A/t)^{1/2}\,(1 - t/\tau_A)^{-3/2}$$

The corresponding plot for x_0 and $\dot{x}_0$ is:



The separatrix distance x_0 becomes infinite at $t = t_A$ because all of the flux passing through the surface has reconnected by this time. The separatrix speed, $\dot{x}_0$, is infinite at both $t = 0$ and $t = 1.0$ when $B_y(x_0, 0)$ is zero. A minimum separatrix speed of $8/(3\sqrt{3})$ (a/τ_A) occurs at $t = \tau_A/4$.