Problem Set Exercises: Planetary Dynamos

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1. Dynamo scaling laws help to predict magnetic field properties from the physical characteristics of a planet. A fairly simple one is known as the magnetostrophic balance scaling and predicts that the magnetic field strength is determined by a balance between Lorentz and Coriolis forces. This means that in a planetary core, the dominant force balance is:

$$2\rho\vec{\Omega} \times \vec{u} \approx \vec{J} \times \vec{B} \tag{1}$$

where ρ is the fluid density, Ω is the angular rotation rate, $\vec{J}$ is current density, $\vec{B}$ is magnetic field and $\vec{u}$ is velocity field.

   (a) Using Ampere's law: $J = \mu^{-1}\nabla \times \vec{B}$ to write J in terms of B in the force balance, and characteristic scales U for velocity, L for length and B for magnetic field, do an order of magnitude calculation to find the magnetic field strength in terms of ρ, Ω, μ, U, and L

   (b) Now use the definition of the Magnetic Reynolds Number: $Re_m = \sigma\mu UL$ where σ is electrical conductivity, to express the magnetostrophic B estimate in terms of ρ, Ω, σ, and Re_m.

   (c) Calculate the magnetostrophic B estimate for Earth and Mercury assuming $Re_m = 1000$. You can use $\sigma = 10^6$ S/m and $\rho = 10^4$ kg/m^3 in your estimates. You will have to look up the rotation rates.

   (d) Now compare this value to the observed field strength for Earth and Mercury. Remember, the observed field strengths are those at the surface whereas the magnetostrophic estimate is in the planetary core. If the magnetic field is dipolar-dominated, then you can scale the field strength from the surface to find the core strength using

$$\frac{B_{cmb}}{B_{surf}} \approx \left(\frac{r_{surf}}{r_{cmb}}\right)^3. \tag{2}$$

   Do your estimates match the field strengths for Earth and Mercury? (remember this was all order of magnitude, so your answers don't have to be exact).

2. In lecture I mention that smaller bodies cool faster through conduction than larger bodies. Here you will demonstrate this with a simple scaling analysis. A body cooling solely through conduction obeys the equation:

$$\frac{\partial T}{\partial t} = \kappa\nabla^2\vec{T} \tag{3}$$

where T is temperature and κ is the thermal diffusivity.

   (a) Using characteristic scales τ for time and L for length, use an order of magnitude argument to determine the characteristic cooling time for a body in terms of its size and κ.

   (b) Using your estimate, determine how much faster a 100 km body cools by conduction compared to a 1000 km body, assuming the same κ for both. I'm looking for the ratio τ_1/τ_2 in this case.

3. The magnetic induction equation (MIE) is given by:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \lambda \nabla^2 \vec{B} \tag{4}$$

As is typical for many PDEs, analytical solutions are rare. Here we will set up a case that is solvable. Consider a perfect conductor (i.e. $\lambda = 0$) with a constant applied vertical magnetic field ($\vec{B} = B_0 \hat{z}$) and a simple horizontal shear flow ($\vec{u} = \frac{Uz}{L}\hat{x}$) where z is the vertical coordinate, x is the horizontal coordinate and U and L are constants. A sketch of the set up is shown in Figure 2.

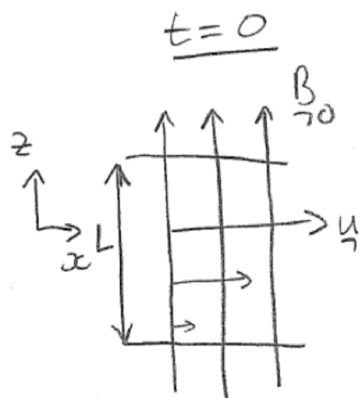

Figure 1: Set up for question 1.

(a) Considering a perfectly conducting fluid, sketch what the magnetic field lines will look like some time later. Probably the easiest way to do this is to imagine a vertical line of fluid particles attached to one of the magnetic field lines, consider how they will be displaced by the flow after time t and then trace out the resulting field line (considering it is frozen into the fluid).

(b) Next we will work through the steps of solving the magnetic induction equation for this case. Plug in the expressions for $\vec{B}_0$ and $\vec{u}$ into the RHS of the MIE for a perfect conductor.

(c) Solve the resulting ODE for the generated field $\vec{B}$ (it should be easy to solve).

(d) You might think you are done, but there is one more important step. Your newly generated field may now be acted upon by the original velocity field to generate even newer field! So now plug in your expression for your new $\vec{B}$ into your original ODE and show that for this example, no newer component of the field is generated. (*Note: For examples with more complicated flows and initial fields, this iterative process can go on indefinitely which is why there aren't tons of analytic solutions for MIE examples, even in the absence of the diffusion term.*)

(e) Compare your analytic solution with your sketch above. Are they consistent?