Creation and destruction of magnetic fields

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Magnetic fields in the Universe

- **Earth**
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales ($10^3$ years)

- **Mercury, Ganymede, (Io), Jupiter, Saturn, Uranus, Neptune** have large scale fields

- **Sun**
  - Magnetic fields from smallest observable scales to size of sun
  - 11 year cycle of large scale field ([Movie](#))
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)

- **Other stars**
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: most likely primordial fields

- **Galaxies**
  - Field structure coupled to observed matter distribution (e.g. spirals)
  - Only dynamo that is directly observable

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Scope of this lecture

- Processes of magnetic field generation and destruction in turbulent plasma flows
- Introduction to general concepts of dynamo theory (this is not a lecture about the solar dynamo!)

Outline

- MHD, induction equation
- Some general remarks and definitions regarding dynamos
- Large scale dynamos (mean field theory)
  - Kinematic theory
  - Characterization of possible dynamos
  - Non-kinematic effects
- 3D simulations
MHD equations

The full set of MHD equations combines the induction equation with the Navier-Stokes equations including the Lorentz-force:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})
\]

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla p + \rho \mathbf{g} + \frac{1}{\mu_0} \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} + \nabla \cdot \tau
\]

\[
\frac{\partial e}{\partial t} = -\rho (\mathbf{v} \cdot \nabla) e - p \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T) + Q_\nu + Q_\eta
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})
\]

Assumptions:

- Validity of continuum approximation (enough particles to define averages)
- Non-relativistic motions, low frequencies
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure
MHD equations

Viscous stress tensor $\tau$

$$\Lambda_{ik} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$$

$$\tau_{ik} = 2 \rho \nu \left( \Lambda_{ik} - \frac{1}{3} \delta_{ik} \nabla \cdot \mathbf{v} \right)$$

$$Q_\nu = \tau_{ik} \Lambda_{ik} ,$$

Ohmic dissipation $Q_\eta$

$$Q_\eta = \frac{\eta}{\mu_0} (\nabla \times \mathbf{B})^2 .$$

Equation of state

$$p = \frac{\rho e}{\gamma - 1} .$$

$\nu$, $\eta$ and $\kappa$: viscosity, magnetic diffusivity and thermal conductivity

$\mu_0$ denotes the permeability of vacuum
Kinematic approach

- Solving the 3D MHD equations is not always feasible
- Semi-analytical approach preferred for understanding fundamental properties of dynamos
- Evaluate turbulent induction effects based on induction equation for a given velocity field
  - Velocity field assumed to be given as 'background' turbulence, Lorentz-force feedback neglected (sufficiently weak magnetic field)
  - What correlations of a turbulent velocity field are required for dynamo (large scale) action?
  - Theory of onset of dynamo action, but not for non-linear saturation

- More detailed discussion of induction equation
Ohm’s law

Equation of motion for drift velocity $v_d$ of electrons

$$m_e \left( \frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}} \right) = -e(E + v_d \times B) - \nabla p_e$$

$\tau_{ei}$: collision time between electrons and ions
$-e$: electron charge
$m_e$: electron mass
$p_e$: electron pressure

With the electric current: $j = -nev_d$ this gives the generalized Ohm’s law:

$$\frac{\partial j}{\partial t} + \frac{j}{\tau_{ei}} = \frac{n_e e^2}{m_e} E - \frac{e}{m_e} j \times B + \frac{n_e e}{m_e} \nabla p_e$$

Simplifications:

- $\tau_{ei} \omega_L \ll 1$, $\omega_L = eB/m_e$: Larmor frequency
- neglect $\nabla p_e$
- low frequencies (no plasma oscillations)
Simplified Ohm’s law

\[ j = \sigma E \]

with the plasma conductivity

\[ \sigma = \frac{\tau_{ei} n_e e^2}{m_e} \]

The Ohm’s law we derived so far is only valid in the co-moving frame of the plasma. Under the assumption of non-relativistic motions this transforms in the laboratory frame to

\[ j = \sigma (E + v \times B) \]
Using Ampere’s law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ yields for the electric field in the laboratory frame

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B}$$

leading to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma}.$$
Advection, diffusion, magnetic Reynolds number

$L$: typical length scale  
$U$: typical velocity scale  
$L/U$: time unit

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B} - \frac{1}{R_m} \nabla \times \mathbf{B} \right)
\]

with the magnetic Reynolds number

\[
R_m = \frac{U L}{\eta}.
\]

$R_m \ll 1$: diffusion dominated regime

\[
\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B}.
\]

Only decaying solutions with decay (diffusion) time scale

\[
\tau_d \sim \frac{L^2}{\eta}.
\]
Advection, diffusion, magnetic Reynolds number

\[ R_m \gg 1 \text{ advection dominated regime (ideal MHD)} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]

Equivalent expression

\[ \frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B} \nabla \cdot \mathbf{v} \]

- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression
Advection, diffusion, magnetic Reynolds number

<table>
<thead>
<tr>
<th>Object</th>
<th>$\eta$ [m$^2$/s]</th>
<th>$L$ [m]</th>
<th>$U$ [m/s]</th>
<th>$R_m$</th>
<th>$\tau_d$</th>
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</thead>
<tbody>
<tr>
<td>earth (outer core)</td>
<td>2</td>
<td>$10^6$</td>
<td>$10^{-3}$</td>
<td>300</td>
<td>$10^4$ years</td>
</tr>
<tr>
<td>sun (plasma conductivity)</td>
<td>1</td>
<td>$10^8$</td>
<td>100</td>
<td>$10^{10}$</td>
<td>$10^9$ years</td>
</tr>
<tr>
<td>sun (turbulent conductivity)</td>
<td>$10^8$</td>
<td>$10^8$</td>
<td>100</td>
<td>100</td>
<td>3 years</td>
</tr>
<tr>
<td>liquid sodium lab experiment</td>
<td>0.1</td>
<td>1</td>
<td>10</td>
<td>100</td>
<td>10 s</td>
</tr>
</tbody>
</table>

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Creation and destruction of magnetic fields
Alfvén’s theorem

Let $\Phi$ be the magnetic flux through a surface $F$ with the property that its boundary $\partial F$ is moving with the fluid:

$$\Phi = \int_F \mathbf{B} \cdot d\mathbf{f} \rightarrow \frac{d\Phi}{dt} = 0$$

- Flux is ‘frozen’ into the fluid
- Field lines ‘move’ with plasma
Dynamos: Motivation

- For $v = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$

- **Earth and other planets:**
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\text{Curie}}$ and field highly variable $\rightarrow$ field must be maintained by active process

- **Sun and other stars:**
  - Evidence for solar magnetic field for $\sim 300\,000$ years ($^{10}\text{Be}$)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale $\sim 10$ years (turbulent diffusivity)
Mathematical definition of dynamo

A bounded volume with the surface $\partial S$, $\mathbf{B}$ maintained by currents contained within $S$

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) \quad \text{in } S
\]

\[
\nabla \times \mathbf{B} = 0 \quad \text{outside } S
\]

\[
[B] = 0 \quad \text{across } \partial S
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

$\mathbf{v} = 0$ outside $S$, $\mathbf{n} \cdot \mathbf{v} = 0$ on $\partial S$ and

\[
E_{\text{kin}} = \int_S \frac{1}{2} \rho \mathbf{v}^2 \, dV \leq E_{\text{max}} \quad \forall \ t
\]

$\mathbf{v}$ is a dynamo if an initial condition $\mathbf{B} = \mathbf{B}_0$ exists so that

\[
E_{\text{mag}} = \int_{-\infty}^{\infty} \frac{1}{2 \mu_0} \mathbf{B}^2 \, dV \geq E_{\text{min}} \quad \forall \ t
\]
Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) \( \mathbf{B} = \mathbf{B} + \mathbf{B}' \):

\[
E_{\text{mag}} = \int \frac{1}{2\mu_0} \mathbf{B}^2 \, dV + \int \frac{1}{2\mu_0} \mathbf{B}'^2 \, dV.
\]

- **Small scale dynamo**: \( \mathbf{B}^2 \ll \mathbf{B}'^2 \)
- **Large scale dynamo**: \( \mathbf{B}^2 \geq \mathbf{B}'^2 \)

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large \( R_m \), large scale dynamos require additional large scale symmetries (see second half of this lecture).
Amplification through field line stretching
Twist-fold required to repack field into original volume
Magnetic diffusivity allows for change of topology
Influence of magnetic diffusivity on growth rate

- **Fast dynamo**: growth rate independent of $R_m$ (stretch-twist-fold mechanism)
- **Slow dynamo**: growth rate limited by resistivity (stretch-reconnect-repack)

- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast
Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

\[ \mathbf{B} = B \mathbf{e}_\Phi + \nabla \times (A \mathbf{e}_\Phi) \]
\[ \mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + \Omega r \sin \theta \mathbf{e}_\Phi \]

Differential rotation most dominant shear flow in stellar convection zones:

Meridional flow by-product of DR, observed as poleward surface flow in case of the sun
Differential rotation and meridional flow

Spherical geometry:

\[
\frac{\partial B}{\partial t} + \frac{1}{r} \left( \frac{\partial}{\partial r} (rv_r B) + \frac{\partial}{\partial \theta} (v_\theta B) \right) =
\]

\[
r \sin \theta B_p \cdot \nabla \Omega + \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) B
\]

\[
\frac{\partial A}{\partial t} + \frac{1}{r \sin \theta} v_p \cdot \nabla (r \sin \theta A) = \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) A
\]

- **Meridional flow:** Independent advection of poloidal and toroidal field
- **Differential rotation:** Source for toroidal field (if poloidal field not zero)
- **Diffusion:** Sink for poloidal and toroidal field
- **No term capable of maintaining poloidal field against Ohmic decay!**
Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field
Cowling’s anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.

Ohm’s law of the form $\mathbf{j} = \sigma \mathbf{E}$ only decaying solutions, focus here on $\mathbf{j} = \sigma (\mathbf{v} \times \mathbf{B})$.

On O-type neutral line $\mathbf{B}_p$ is zero, but $\mu_0 \mathbf{j}_t = \nabla \times \mathbf{B}_p$ has finite value, but cannot be maintained by $(\mathbf{v} \times \mathbf{B})_t = (\mathbf{v}_p \times \mathbf{B}_p)$. 
Some history:

- 1919 Sir Joeseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling’s anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical frame work of mean field theory developed
- last 2 decades 3D dynamo simulations
Reynolds rules

We need to define an averaging procedure to define the mean and the fluctuating field.
For any function $f$ and $g$ decomposed as $f = \bar{f} + f'$ and $g = \bar{g} + g'$ we require that the Reynolds rules apply

\[ \bar{f} = \bar{f} \rightarrow f' = 0 \]
\[ \bar{f} + \bar{g} = \bar{f} + \bar{g} \]
\[ f \bar{g}' = \bar{f} \bar{g} \rightarrow f' \bar{g} = 0 \]
\[ \frac{\partial f}{\partial x_i} = \frac{\partial \bar{f}}{\partial x_i} \]
\[ \frac{\partial f}{\partial t} = \frac{\partial \bar{f}}{\partial t} . \]

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)
Meanfield induction equation

Average of induction equation:

\[
\frac{\partial \overline{B}}{\partial t} = \nabla \times \left( \overline{v'} \times \overline{B'} + \overline{v} \times \overline{B} - \eta \nabla \times \overline{B} \right)
\]

New term resulting from small scale effects:

\[
\overline{\mathcal{E}} = \overline{v'} \times \overline{B'}
\]

Fluctuating part of induction equation:

\[
\left( \frac{\partial}{\partial t} - \eta \Delta \right) \overline{B'} - \nabla \times (\overline{v} \times \overline{B'}) = \nabla \times \left( \overline{v}' \times \overline{B} + \overline{v}' \times \overline{B'} - \overline{v'} \times \overline{B'} \right)
\]

Kinematic approach: \( \overline{v'} \) assumed to be given

- Solve for \( \overline{B'} \), compute \( \overline{v'} \times \overline{B'} \) and solve for \( \overline{B} \)
- Term \( \overline{v'} \times \overline{B'} - \overline{v'} \times \overline{B'} \) leading to higher order correlations (closure problem)
Second order correlation approximation (SOCA)

Second order term can be neglected if

- $|B'| \ll |\bar{B}|$
- $|v' \times B' - \bar{v}' \times \bar{B}'| \ll |v' \times \bar{B}|$
- $\nabla \times (v' \times B' - \bar{v}' \times \bar{B}')$ correlates only weakly with $v'$

Sufficient condition:

- $R_m \ll 1$ or $S = v \tau_c / l_c \ll 1 \quad \implies \quad |B'| \ll |\bar{B}|$
- Problem: $R_m \gg 1$ and $S \sim 1$ in stellar convection zones

In praxis it works better than it should!
Second order correlation approximation (SOCA)

Neglecting higher order moments and assume $\bar{\tau} \gg \tau_c$:

$$B' \approx \tau_c \nabla \times (v' \times \bar{B}) = -\tau_c (v' \cdot \nabla) \bar{B} + \tau_c [(\bar{B} \cdot \nabla)v' - \bar{B} \nabla \cdot v']$$

leads to the expression:

$$\bar{E} = \alpha \bar{B} + \gamma \times \bar{B} - \beta \nabla \times \bar{B} + \ldots$$

with ($\alpha$ and $\beta$ are symmetric tensors):

$$\alpha_{ij} = \frac{1}{2} \tau_c \left( \varepsilon_{ikl} v'_{k} \frac{\partial v'_l}{\partial x_j} + \varepsilon_{jkl} v'_{k} \frac{\partial v'_l}{\partial x_i} \right)$$

$$\gamma_i = -\frac{1}{2} \tau_c \frac{\partial}{\partial x_k} v'_i v'_k$$

$$\beta_{ij} = \frac{1}{2} \tau_c \left( v'^2 \delta_{ij} - v'_i v'_j \right)$$
Second order correlation approximation (SOCA)

Simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence:

\[ v_i' v_j' \sim \delta_{ij} , \quad \alpha_{ij} = \alpha \delta_{ij} , \quad \beta_{ij} = \eta_t \delta_{ij} \]

Leads to:

\[
\begin{align*}
\alpha & = \frac{1}{3} \alpha_{ii} = -\frac{1}{3} \tau_c \bar{v}' \cdot (\nabla \times \bar{v}') \sim \frac{\eta_t}{l_c} \sim v_{\text{rms}} \\
\eta_t & = \frac{1}{3} \beta_{ii} = \frac{1}{3} \tau_c \bar{v}'^2 \sim l_c v_{\text{rms}} \\
\gamma & = -\frac{1}{2} \nabla \eta_t
\end{align*}
\]

Induction equation for \( \bar{B} \):

\[
\frac{\partial \bar{B}}{\partial t} = \nabla \times \left[ \alpha \bar{B} + (\bar{v} + \gamma) \times \bar{B} - (\eta + \eta_t) \nabla \times \bar{B} \right]
\]
Turbulent diffusivity - destruction of magnetic field

Turbulent diffusivity dominant dissipation process for large scale field in case of large $R_m$:

$$\eta_t = \frac{1}{3} \tau_c \overline{v'^2} \sim L \nu_{\text{rms}} \sim R_m \eta \gg \eta$$

- Formally $\eta_t$ comes from advection term (transport term, non-dissipative)
- Turbulent cascade transporting magnetic energy from the large scale $L$ to the micro scale $l_m$ (advection + reconnection)

$$\eta j_m^2 \sim \eta_t j^2 \rightarrow \frac{B_m}{l_m} \sim \sqrt{R_m \overline{B}}$$

**Important**: The large scale determines the energy dissipation rate, $l$ adjusts to allow for the dissipation on the microscale.

Present for isotropic homogeneous turbulence
Turbulent diamagnetism, turbulent pumping

Expulsion of flux from regions with larger turbulence intensity 'diamagnetism'

\[ \gamma = -\frac{1}{2} \nabla \eta_t \]

Downward directed at base of convection zone

Turbulent pumping (stratified convection):
- Upflows expand, downflows converge
- Stronger velocity and smaller filling factor of downflows
- Mean advection effect of up- and downflows does not cancel
- Downward transport found in numerical simulations

Requires inhomogeneity (stratification)
Kinematic $\alpha$-effect

\[ \alpha = -\frac{1}{3} \tau_c \mathbf{v}' \cdot (\nabla \times \mathbf{v}') \quad H_k = \mathbf{v}' \cdot (\nabla \times \mathbf{v}') \text{ kinetic helicity} \]

Requires rotation + additional preferred direction (stratification)
Turbulent induction effects require reconnection to operate; however, the expressions

\[ \alpha_{ij} = \frac{1}{2} \tau c \left( \varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_j} + \varepsilon_{jkl} v'_k \frac{\partial v'_l}{\partial x_i} \right) \]

\[ \gamma_i = -\frac{1}{2} \tau c \frac{\partial}{\partial x_k} v'_i v'_k \]

\[ \beta_{ij} = \frac{1}{2} \tau c \left( \overline{v'^2} \delta_{ij} - v'_i v'_j \right) \]

are independent of \( \eta \) (in this approximation), indicating fast dynamo action.
Validity of Mean field expansion

Second order correlation approximation:
- At best marginally justified
- Works better than it should

Most general form for mean field expansion:

$$\overline{E}_i(x, t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' K_{ij}(x, t, x', t') \overline{B}_j(x', t') .$$

Sufficient scale separation
- $l_c \ll L$
- $\tau_c \ll \tau_L$

leads to:

$$\overline{E} = \alpha \overline{B} + \gamma \times \overline{B} - \beta \nabla \times \overline{B} - \delta \times \nabla \times \overline{B} + \ldots$$

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Large scale convection (M. Miesch, HAO)
Symmetry constraints

\(\alpha, \beta, \gamma\) and \(\delta\) depend on large scale symmetries of the system defining the symmetry properties of the turbulence (e.g. rotation and stratification). Additional to that the expansion

\[
\overline{\mathbf{E}} = \alpha \overline{\mathbf{B}} + \gamma \times \overline{\mathbf{B}} - \beta \nabla \times \overline{\mathbf{B}} - \delta \times \nabla \times \overline{\mathbf{B}} + \ldots
\]

is a relation between polar and axial vectors:

- \(\overline{\mathbf{E}}\): polar vector, independent from handedness of coordinate system
- \(\overline{\mathbf{B}}\): axial vector, involves handedness of coordinate system in definition (curl operator, cross product)

Handedness of coordinate system pure convention (contains no physics), consistency requires:

- \(\alpha, \delta\): pseudo tensor
- \(\beta, \gamma\): true tensors
Symmetry constraints

Turbulence with rotation and stratification

- true tensors: \( \delta_{ij}, g_i, g_i g_j, \Omega_i \Omega_j, \Omega_i \varepsilon_{ijk} \)
- pseudo tensors: \( \varepsilon_{ijk}, \Omega_i, \Omega_i g_j, g_i \varepsilon_{ijk} \)

Symmetry constraints allow only certain combinations:

\[
\alpha_{ij} = \alpha_0 (g \cdot \Omega) \delta_{ij} + \alpha_1 (g_i \Omega_j + g_j \Omega_i), \quad \gamma_i = \gamma_0 g_i + \gamma_1 \varepsilon_{ijk} g_j \Omega_k
\]

\[
\beta_{ij} = \beta_0 \delta_{ij} + \beta_1 g_i g_j + \beta_2 \Omega_i \Omega_j, \quad \delta_i = \delta_0 \Omega_i
\]

The scalars \( \alpha_0 \ldots \delta_0 \) depend on quantities of the turbulence such as rms velocity and correlation times scale.

- isotropic turbulence: only \( \beta \)
- + stratification: \( \beta + \gamma \)
- + rotation: \( \beta + \delta \)
- + stratification + rotation: \( \alpha \) can exist
What is needed to circumvent Cowling’s theorem?

- Crucial for Cowling’s theorem: Impossibility to drive a current parallel to magnetic field
- Cowling’s theorem does not apply to mean field if a mean current can flow parallel to the mean field (since total field non-axisymmetric this is not a contradiction!)

\[
\mathbf{j} = \tilde{\sigma} \left( \overline{E} + \overline{\nu} \times \overline{B} + \gamma \times \overline{B} + \alpha \overline{B} \right)
\]

\(\tilde{\sigma}\) contains contributions from \(\eta, \beta\) and \(\delta\).

Ways to circumvent Cowling:

- \(\alpha\)-effect
- anisotropic conductivity (off diagonal elements + \(\delta\)-effect)
\( \alpha^2 \)-dynamo

**Induction of field parallel to current (producing helical field!)**

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) = \alpha \mu_0 \mathbf{j}
\]

**Dynamo cycle:**

\[
\mathbf{B}_t \overset{\alpha}{\rightarrow} \mathbf{B}_p \overset{\alpha}{\rightarrow} \mathbf{B}_t
\]

- Poloidal and toroidal field of similar strength
- In general stationary solutions
Dynamo cycle:

\[ B_t \xrightarrow{\alpha} B_p \xrightarrow{\Omega} B_t \]

- Toroidal field much stronger than poloidal field
- In general traveling (along lines of constant \( \Omega \)) and periodic solutions
\[ \frac{\partial B}{\partial t} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r v_r B) + \frac{\partial}{\partial \theta} (v_\theta B) \right) = r \sin B_p \cdot \nabla \Omega \]
\[ + \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) B \]
\[ \frac{\partial A}{\partial t} + \frac{1}{r \sin \theta} v_p \cdot \nabla (r \sin \theta A) = \alpha B + \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) A \]

- Dimensionless measure for strength of \( \Omega- \) and \( \alpha-\)effect

\[ D_\Omega = \frac{R^2 \Delta \Omega}{\eta_t} \quad D_\alpha = \frac{R \alpha}{\eta_t} \]

- Dynamo excited if dynamo number

\[ D = D_\Omega D_\alpha > D_{crit} \]

Movie: \( \alpha\Omega\)-dynamo
Meridional flow:
- Poleward at top of convection zone
- Equatorward at bottom of convection zone

Effect of advection:
- Equatorward propagation of activity
- Correct phase relation between poloidal and toroidal field
- Circulation time scale of flow sets dynamo period
- **Requirement:** Sufficiently low turbulent diffusivity

**Movie:** Flux-transport-dynamo (M. Dikpati, HAO)
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \delta \times (\nabla \times \mathbf{B}) \right] \sim \nabla \times (\Omega \times \mathbf{j}) \sim \frac{\partial \mathbf{j}}{\partial z} \]

- similar to \( \alpha \)-effect, but additional \( z \)-derivative of current
- couples poloidal and toroidal field
- \( \delta^2 \) dynamo is not possible:
  \[ \mathbf{j} \cdot \mathbf{E} = \mathbf{j} \cdot (\delta \times \mathbf{j}) = 0 \]
- \( \delta \)-effect is controversial (not all approximations give a non-zero effect)
- in most situations \( \alpha \) dominates
Dynamos and magnetic helicity

Magnetic helicity (integral measure of field topology):

\[ H_m = \int \mathbf{A} \cdot \mathbf{B} \, dV \]

has following conservation law (no helicity fluxes across boundaries):

\[ \frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, dV = -2\mu_0 \eta \int \mathbf{j} \cdot \mathbf{B} \, dV \]

Decomposition into small and large scale part:

\[ \frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, dV = +2 \int \mathbf{E} \cdot \mathbf{B} \, dV - 2\mu_0 \eta \int \mathbf{j} \cdot \mathbf{B} \, dV \]

\[ \frac{d}{dt} \int \mathbf{A}' \cdot \mathbf{B}' \, dV = -2 \int \mathbf{E} \cdot \mathbf{B} \, dV - 2\mu_0 \eta \int \mathbf{j}' \cdot \mathbf{B}' \, dV \]
Dynamos and magnetic helicity

Dynamos have helical fields:

- $\alpha$-effect induces magnetic helicity of same sign on large scale
- $\alpha$-effect induces magnetic helicity of opposite sign on small scale

Saturation process (on scale $\sim R_m\tau_c$):

$$
\bar{j}' \cdot \bar{B}' = -\bar{j} \cdot \bar{B} \rightarrow \frac{|\bar{B}|}{|\bar{B}'|} \sim \sqrt{\frac{L}{l}}
$$

$$
\bar{j}' \cdot \bar{B}' = -\frac{\alpha \bar{B}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{j} \cdot \bar{B}
$$

Time scales:

- Galaxy: $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- Sun: $\sim 10^8$ years
- Earth: $\sim 10^6$ years
Non-kinematic effects

Proper way to treat them: 3D simulations

- Still very challenging
- Has been successful for geodynamo, but not for solar dynamo

Semi-analytical treatment of Lorentz-force feedback in mean field models:

- Macroscopic feedback: Change of the mean flow (differential rotation, meridional flow) through the mean Lorentz-force

\[ \bar{f} = \bar{j} \times \bar{B} + \bar{j}' \times \bar{B}' \]

- Mean field model including mean field representation of full MHD equations:
  Movie: Non-kinematic flux-transport dynamo

- Microscopic feedback: Change of turbulent induction effects (e.g. \( \alpha \)-quenching)
Feedback of Lorentz force on small scale motions:

- Intensity of turbulent motions significantly reduced if
  \[ \frac{1}{2\mu_0} B^2 > \frac{1}{2} \varrho v_{rms}^2. \]  
  Typical expression used

\[ \alpha = \frac{\alpha_k}{1 + \frac{B^2}{B_{eq}^2}} \]

with the equipartition field strength \( B_{eq} = \sqrt{\mu_0 \varrho v_{rms}} \)

- Similar quenching also expected for turbulent diffusivity
- Additional quenching of \( \alpha \) due to topological constraints possible (helicity conservation)

Controversial!
Symmetry of momentum and induction equation $\mathbf{v}' \leftrightarrow \mathbf{B}'$:

$$\frac{d\mathbf{v}'}{dt} = \frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla) \mathbf{B}' + \ldots$$

$$\frac{d\mathbf{B}'}{dt} = (\mathbf{B} \cdot \nabla) \mathbf{v}' + \ldots$$

$$\mathcal{E} = \mathbf{v}' \times \mathbf{B}'$$

Strongly motivates magnetic term for $\alpha$-effect (Pouquet et al. 1976):

$$\alpha = \frac{1}{3} \tau_c \left( \frac{1}{\rho} \mathbf{j}' \cdot \mathbf{B}' - \mathbf{\omega}' \cdot \mathbf{v}' \right)$$

- **Kinetic $\alpha$:** $\overline{\mathbf{B}} + \mathbf{v}' \rightarrow \mathbf{B}' \rightarrow \mathbf{E}$
- **Magnetic $\alpha$:** $\overline{\mathbf{B}} + \mathbf{B}' \rightarrow \mathbf{v}' \rightarrow \mathbf{E}$
From helicity conservation one expects

$$\mathbf{j}' \cdot \mathbf{B}' \sim -\alpha \mathbf{B}^2$$

leading to algebraic quenching

$$\alpha = \frac{\alpha_k}{1 + g \frac{B^2}{B_{eq}^2}}$$

With the asymptotic expression (steady state)

$$\mathbf{j}' \cdot \mathbf{B}' = -\frac{\alpha \mathbf{B}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \mathbf{j} \cdot \mathbf{B}$$

we get

$$\alpha = \frac{\alpha_k + \frac{\eta_t^2}{\eta} \frac{\mu_0 \mathbf{j} \cdot \mathbf{B}}{B_{eq}^2}}{1 + \frac{\eta_t}{\eta} \frac{B^2}{B_{eq}^2}}$$
Microscopic feedback

Catastrophic $\alpha$-quenching ($R_m \gg 1$!) in case of steady state and homogeneous $\overline{B}$:

$$ \alpha = \frac{\alpha_k}{1 + R_m \frac{\overline{B}^2}{B_{eq}^2}} $$

If $\overline{j} \cdot \overline{B} \neq 0$ (dynamo generated field) and $\eta_t$ unquenched:

$$ \alpha \approx \eta_t \mu_0 \frac{\overline{j} \cdot \overline{B}}{\overline{B}^2} \sim \frac{\eta_t}{L} \sim \frac{\eta_t}{L} \frac{l_c}{L} \sim \alpha_k \frac{l_c}{L} $$

- In general $\alpha$-quenching dynamic process: linked to time evolution of helicity
- Boundary conditions matter: Loss of small scale current helicity can alleviate catastrophic quenching
- Catastrophic $\alpha$-quenching turns large scale dynamo into slow dynamo
Stationary state reached on time scale $R_m \tau_c$:

- **Galaxy:** $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- **Sun:** $\sim 10^8$ years
- **Earth:** $\sim 10^6$ years
- **Universe too young for galaxies to worry about stationary state!**
- **Sun, geodynamo had enough time too saturate**
  - Sun: Possibility that helicity loss through photosphere alleviates quenching
  - Geodynamo: $R_m \sim 300$ not that catastrophic?
- **No observational evidence for catastrophic $\alpha$-quenching, but fundamental question for theory!**
3D simulations

Why not just solving the full system to account for all non-linear effects?

- Most systems have $R_e \gg R_m \gg 1$, requiring high resolution
- Large scale dynamos evolve on time scales $\tau_c \ll t \ll \tau_\eta$, requiring long runs compared to convective turn over
- 3D simulations successful for geodynamo
  - $R_m \sim 300$: all relevant magnetic scales resolvable
  - Incompressible system
- Solar dynamo: Ingredients can be simulated
  - Compressible system: density changes by $10^6$ through convection zone
  - Boundary layer effects: Tachocline, difficult to simulate (strongly subadiabatic stratification, large time scales)
  - Magnetic structures down to 1000 km most likely important
  - Evolve $5000^3$ box over 1000 $\tau_c$!
  - Small scale dynamos can be simulated (for $P_m \sim 1$)
Summarizing remarks

Destruction of magnetic field:

- Turbulent diffusivity: cascade of magnetic energy from large scale to dissipation scale (advection+reconnection)
- Enhances dissipation of large field by a factor $R_m$

Creation of magnetic field:

- Small scale dynamo (non-helical)
  - Amplification of field on and below scale of turbulence
  - Stretch-twist-fold-(reconnect)
  - Produces non-helical field and does not require helical motions
  - Current research: behavior for $P_m \ll 1$

- Large scale dynamo (helical)
  - Amplification of field on scales larger than scale of turbulence
  - Produces helical field and does require helical motions
  - Requires rotation + additional symmetry direction
    (controversial $\Omega \times J$ effect does not require helical motions)
  - Current research: catastrophic vs. non-catastrophic quenching