Creation and destruction of magnetic fields

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Magnetic fields in the Universe

- **Earth**
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales ($10^3$ years)
- **Mercury, Ganymede, (Io), Jupiter, Saturn, Uranus, Neptune** have large scale fields
- **Sun**
  - Magnetic fields from smallest observable scales to size of sun
  - 11 year cycle of large scale field ([Movie](#))
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)
- **Other stars**
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: most likely primordial fields
- **Galaxies**
  - Field structure coupled to observed matter distribution (e.g. spirals)
  - Only dynamo that is directly observable
Scope of this lecture

- Processes of magnetic field generation and destruction in turbulent plasma flows
- Introduction to general concepts of dynamo theory (this is not a lecture about the solar dynamo!)
- Outline
  - MHD, induction equation
  - Some general remarks and definitions regarding dynamos
  - Large scale dynamos (mean field theory)
    - Kinematic theory
    - Characterization of possible dynamos
    - Non-kinematic effects
  - 3D simulations
MHD equations

The full set of MHD equations combines the induction equation with the Navier-Stokes equations including the Lorentz-force:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{v}) \\
\rho \frac{\partial \mathbf{v}}{\partial t} &= -\rho (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla p + \rho \mathbf{g} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \mathbf{\tau} \\
\rho \frac{\partial e}{\partial t} &= -\rho (\mathbf{v} \cdot \nabla) e - \rho \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T) + Q_\nu + Q_\eta \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})
\end{align*}
\]

Assumptions:
- Validity of continuum approximation (enough particles to define averages)
- Non-relativistic motions, low frequencies
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure
MHD equations

Viscous stress tensor $\tau$

$$\Lambda_{ik} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$$

$$\tau_{ik} = 2 \rho \nu \left( \Lambda_{ik} - \frac{1}{3} \delta_{ik} \nabla \cdot \mathbf{v} \right)$$

$$Q_\nu = \tau_{ik} \Lambda_{ik} ,$$

Ohmic dissipation $Q_\eta$

$$Q_\eta = \frac{\eta}{\mu_0} (\nabla \times \mathbf{B})^2 .$$

Equation of state

$$p = \frac{\rho e}{\gamma - 1} .$$

$\nu, \eta$ and $\kappa$: viscosity, magnetic diffusivity and thermal conductivity

$\mu_0$ denotes the permeability of vacuum
Kinematic approach

- Solving the 3D MHD equations is not always feasible
- Semi-analytical approach preferred for understanding fundamental properties of dynamos
- Evaluate turbulent induction effects based on induction equation for a given velocity field
  - Velocity field assumed to be given as 'background' turbulence, Lorentz-force feedback neglected (sufficiently weak magnetic field)
  - What correlations of a turbulent velocity field are required for dynamo (large scale) action?
  - Theory of onset of dynamo action, but not for non-linear saturation

- More detailed discussion of induction equation
Ohm’s law

Equation of motion for drift velocity $v_d$ of electrons

$$m_e \left( \frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}} \right) = -e(E + v_d \times B) - \nabla p_e$$

$\tau_{ei}$: collision time between electrons and ions
$-e$: electron charge
$m_e$: electron mass
$p_e$: electron pressure

With the electric current: $j = -n_e v_d$ this gives the generalized Ohm’s law:

$$\frac{\partial j}{\partial t} + \frac{j}{\tau_{ei}} = \frac{n_e e^2}{m_e} E - \frac{e}{m_e} j \times B + \frac{n_e e}{m_e} \nabla p_e$$

Simplifications:
- $\tau_{ei} \omega_L \ll 1$, $\omega_L = eB/m_e$: Larmor frequency
- neglect $\nabla p_e$
- low frequencies (no plasma oscillations)
Simplified Ohm’s law

\[ \mathbf{j} = \sigma \mathbf{E} \]

with the plasma conductivity

\[ \sigma = \frac{\tau_{ei} n_e e^2}{m_e} \]

The Ohm’s law we derived so far is only valid in the co-moving frame of the plasma. Under the assumption of non-relativistic motions this transforms in the laboratory frame to

\[ \mathbf{j} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]
Using Ampere’s law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ yields for the electric field in the laboratory frame

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B}$$

leading to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma}.$$
Advection, diffusion, magnetic Reynolds number

$L$: typical length scale  
$U$: typical velocity scale  
$L/U$: time unit

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B} - \frac{1}{R_m} \nabla \times \mathbf{B} \right)
\]

with the magnetic Reynolds number

\[
R_m = \frac{U L}{\eta}.
\]

$R_m \ll 1$: diffusion dominated regime

\[
\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B}.
\]

Only decaying solutions with decay (diffusion) time scale

\[
\tau_d \sim \frac{L^2}{\eta}.
\]
Advection, diffusion, magnetic Reynolds number

\[ R_m \gg 1 \text{ advection dominated regime (ideal MHD)} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]

Equivalent expression

\[ \frac{\partial \mathbf{B}}{\partial t} = - (\mathbf{v} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{v} - \mathbf{B} \nabla \cdot \mathbf{v} \]

- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression
### Advection, diffusion, magnetic Reynolds number

<table>
<thead>
<tr>
<th>Object</th>
<th>$\eta$ [m$^2$/s]</th>
<th>$L$ [m]</th>
<th>$U$ [m/s]</th>
<th>$R_m$</th>
<th>$\tau_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>earth (outer core)</td>
<td>2</td>
<td>$10^6$</td>
<td>$10^{-3}$</td>
<td>300</td>
<td>$10^4$ years</td>
</tr>
<tr>
<td>sun (plasma conductivity)</td>
<td>1</td>
<td>$10^8$</td>
<td>100</td>
<td>$10^{10}$</td>
<td>$10^9$ years</td>
</tr>
<tr>
<td>sun (turbulent conductivity)</td>
<td>$10^8$</td>
<td>$10^8$</td>
<td>100</td>
<td>100</td>
<td>3 years</td>
</tr>
<tr>
<td>liquid sodium lab experiment</td>
<td>0.1</td>
<td>1</td>
<td>10</td>
<td>100</td>
<td>10 s</td>
</tr>
</tbody>
</table>
Alfven’s theorem

Let $\Phi$ be the magnetic flux through a surface $F$ with the property that its boundary $\partial F$ is moving with the fluid:

$$\Phi = \int_F \mathbf{B} \cdot d\mathbf{f} \rightarrow \frac{d\Phi}{dt} = 0$$

- Flux is 'frozen' into the fluid
- Field lines 'move' with plasma
Dynamos: Motivation

- For $v = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$

- **Earth and other planets:**
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\text{Curie}}$ and field highly variable $\rightarrow$ field must be maintained by active process

- **Sun and other stars:**
  - Evidence for solar magnetic field for $\sim 300 000$ years ($^{10}\text{Be}$)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale $\sim 10$ years (turbulent diffusivity)
Mathematical definition of dynamo

$S$ bounded volume with the surface $\partial S$, $\mathbf{B}$ maintained by currents contained within $S$

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) \quad \text{in } S
\]

\[
\nabla \times \mathbf{B} = 0 \quad \text{outside } S
\]

\[
[B] = 0 \quad \text{across } \partial S
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

$v = 0$ outside $S$, $\mathbf{n} \cdot \mathbf{v} = 0$ on $\partial S$ and

\[
E_{\text{kin}} = \int_S \frac{1}{2} \rho v^2 \, dV \leq E_{\text{max}} \quad \forall \, t
\]

$v$ is a dynamo if an initial condition $\mathbf{B} = \mathbf{B}_0$ exists so that

\[
E_{\text{mag}} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \mathbf{B}^2 \, dV \geq E_{\text{min}} \quad \forall \, t
\]
Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) $\mathbf{B} = \mathbf{B} + \mathbf{B}'$:

$$E_{\text{mag}} = \int \frac{1}{2\mu_0} \mathbf{B}^2 \, dV + \int \frac{1}{2\mu_0} \mathbf{B}'^2 \, dV .$$

- **Small scale dynamo**: $\mathbf{B}^2 \ll \mathbf{B}'^2$

- **Large scale dynamo**: $\mathbf{B}^2 \geq \mathbf{B}'^2$

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large $R_m$, large scale dynamos require additional large scale symmetries (see second half of this lecture).
Large scale/small scale dynamos

- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Magnetic diffusivity allows for change of topology
Influence of magnetic diffusivity on growth rate

- **Fast dynamo**: growth rate independent of $R_m$ (stretch-twist-fold mechanism)
- **Slow dynamo**: growth rate limited by resistivity (stretch-reconnect-repack)

Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$

Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast
Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

\[ \mathbf{B} = B e_\Phi + \nabla \times (A e_\Phi) \]
\[ \mathbf{v} = v_r e_r + v_\theta e_\theta + \Omega r \sin \theta e_\Phi \]

Differential rotation most dominant shear flow in stellar convection zones:

Meridional flow by-product of DR, observed as poleward surface flow in case of the sun
Spherical geometry:

\[
\frac{\partial B}{\partial t} + \frac{1}{r} \left( \frac{\partial}{\partial r}(rv_r B) + \frac{\partial}{\partial \theta}(v_\theta B) \right) = r \sin \theta B_p \cdot \nabla \Omega + \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) B
\]

\[
\frac{\partial A}{\partial t} + \frac{1}{r \sin \theta} v_p \cdot \nabla (r \sin \theta A) = \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) A
\]

- **Meridional flow:** Independent advection of poloidal and toroidal field
- **Differential rotation:** Source for toroidal field (if poloidal field not zero)
- **Diffusion:** Sink for poloidal and toroidal field
- **No term capable of maintaining poloidal field against Ohmic decay!**
Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field
Cowling’s anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.

Ohm’s law of the form \( \mathbf{j} = \sigma \mathbf{E} \) only decaying solutions, focus here on \( \mathbf{j} = \sigma (\mathbf{v} \times \mathbf{B}) \).

On O-type neutral line \( \mathbf{B}_p \) is zero, but \( \mu_0 \mathbf{j}_t = \nabla \times \mathbf{B}_p \) has finite value, but cannot be maintained by \((\mathbf{v} \times \mathbf{B})_t = (\mathbf{v}_p \times \mathbf{B}_p)\).
Some history:

- 1919 Sir Joseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling’s anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical framework of mean field theory developed
- last 2 decades 3D dynamo simulations
We need to define an averaging procedure to define the mean and the fluctuating field.

For any function \( f \) and \( g \) decomposed as \( f = \bar{f} + f' \) and \( g = \bar{g} + g' \) we require that the Reynolds rules apply

\[
\bar{f} \quad = \quad \bar{f} \quad \rightarrow \quad f' = 0 \\
\bar{f} + \bar{g} \quad = \quad \bar{f} + \bar{g} \\
\bar{f} \bar{g} \quad = \quad \bar{f} \bar{g} \quad \rightarrow \quad f' \bar{g} = 0 \\
\frac{\partial f}{\partial x_i} \quad = \quad \frac{\partial \bar{f}}{\partial x_i} \\
\frac{\partial f}{\partial t} \quad = \quad \frac{\partial \bar{f}}{\partial t}.
\]

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)
Meanfield induction equation

Average of induction equation:

\[
\frac{\partial \overline{B}}{\partial t} = \nabla \times \left( \overline{v}' \times \overline{B}' + \overline{v} \times \overline{B} - \eta \nabla \times \overline{B} \right)
\]

New term resulting from small scale effects:

\[
\overline{\mathcal{E}} = \overline{v}' \times \overline{B}'
\]

Fluctuating part of induction equation:

\[
\left( \frac{\partial}{\partial t} - \eta \Delta \right) \overline{B}' - \nabla \times (\overline{v} \times \overline{B}') = \nabla \times \left( \overline{v}' \times \overline{B} + \overline{v}' \times \overline{B}' - \overline{v}' \times \overline{B}' \right)
\]

Kinematic approach: \( \overline{v}' \) assumed to be given

- Solve for \( \overline{B}' \), compute \( \overline{v}' \times \overline{B}' \) and solve for \( \overline{B} \)
- Term \( \overline{v}' \times \overline{B}' - \overline{v}' \times \overline{B}' \) leading to higher order correlations (closure problem)
Second order term can be neglected if

- $|\mathbf{B}'| \ll |\overline{\mathbf{B}}|$ 
- $|\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'}| \ll |\mathbf{v}' \times \overline{\mathbf{B}}|$ 
- $\nabla \times (\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'})$ correlates only weakly with $\mathbf{v}'$

Sufficient condition:

- $R_m \ll 1$ or $S = v\tau_c/l_c \ll 1$ $\rightarrow |\mathbf{B}'| \ll |\overline{\mathbf{B}}|$ 
- Problem: $R_m \gg 1$ and $S \sim 1$ in stellar convection zones

In praxis it works better than it should!
Neglecting higher order moments and assume $\bar{\tau} \gg \tau_c$:

$$B' \approx \tau_c \nabla \times (v' \times \bar{B}) = -\tau_c (v' \cdot \nabla) \bar{B} + \tau_c [(\bar{B} \cdot \nabla)v' - \bar{B} \nabla \cdot v']$$

leads to the expression:

$$\bar{\mathcal{E}} = \alpha \bar{B} + \gamma \times \bar{B} - \beta \nabla \times \bar{B} + \ldots$$

with ($\alpha$ and $\beta$ are symmetric tensors):

$$\alpha_{ij} = \frac{1}{2} \tau_c \left( \varepsilon_{ikl} v'_k \frac{\partial v'_l}{\partial x_j} + \varepsilon_{jkl} v'_k \frac{\partial v'_l}{\partial x_i} \right)$$

$$\gamma_i = -\frac{1}{2} \tau_c \frac{\partial}{\partial x_k} v'_i v'_k$$

$$\beta_{ij} = \frac{1}{2} \tau_c \left( v'^2 \delta_{ij} - v'_i v'_j \right)$$

Matthias Rempel
Creation and destruction of magnetic fields
Second order correlation approximation (SOCA)

Simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence:

\[ v_i'v_j' \sim \delta_{ij}, \quad \alpha_{ij} = \alpha \delta_{ij}, \quad \beta_{ij} = \eta_t \delta_{ij} \]

Leads to:

\[
\alpha = \frac{1}{3} \alpha_{ii} = -\frac{1}{3} \tau_c \overline{v'} \cdot (\nabla \times \overline{v'}) \sim \frac{\eta_t}{l_c} \sim v_{\text{rms}}
\]

\[
\eta_t = \frac{1}{3} \beta_{ii} = \frac{1}{3} \tau_c \overline{v'^2} \sim l_c v_{\text{rms}}
\]

\[
\gamma = -\frac{1}{2} \nabla \eta_t
\]

Induction equation for \( \overline{B} \):

\[
\frac{\partial \overline{B}}{\partial t} = \nabla \times \left[ \alpha \overline{B} + (\overline{v} + \gamma) \times \overline{B} - (\eta + \eta_t) \nabla \times \overline{B} \right]
\]
Turbulent diffusivity - destruction of magnetic field

Turbulent diffusivity dominant dissipation process for large scale field in case of large $R_m$:

$$\eta_t = \frac{1}{3} \tau_c \langle \mathbf{v}'^2 \rangle \sim L v_{\text{rms}} \sim R_m \eta \gg \eta$$

- Formally $\eta_t$ comes from advection term (transport term, non-dissipative)
- Turbulent cascade transporting magnetic energy from the large scale $L$ to the micro scale $l_m$ (advection + reconnection)

$$\eta j_m^2 \sim \eta_t j^2 \rightarrow \frac{B_m}{l_m} \sim \sqrt{R_m \frac{B}{L}}$$

**Important:** The large scale determines the energy dissipation rate, $l$ adjusts to allow for the dissipation on the microscale. Present for isotropic homogeneous turbulence
Turbulent diamagnetism, turbulent pumping

Expulsion of flux from regions with larger turbulence intensity 'diamagnetism'

\[ \gamma = -\frac{1}{2} \nabla \eta_t \]

Downward directed at base of convection zone

Turbulent pumping (stratified convection):

- Upflows expand, downflows converge
- Stronger velocity and smaller filling factor of downflows
- Mean advection effect of up- and downflows does not cancel
- Downward transport found in numerical simulations

Requires inhomogeneity (stratification)
Kinematic $\alpha$-effect

$$\alpha = -\frac{1}{3}\tau_c v' \cdot (\nabla \times v') \quad H_k = v' \cdot (\nabla \times v')$$ kinetic helicity

Requires rotation + additional preferred direction (stratification)
Turbulent induction effects require reconnection to operate; however, the expressions

\[ \alpha_{ij} = \frac{1}{2} \tau_c \left( \varepsilon_{ikl} v'_{k} \frac{\partial v'_l}{\partial x_j} + \varepsilon_{jkl} v'_{k} \frac{\partial v'_l}{\partial x_i} \right) \]

\[ \gamma_i = -\frac{1}{2} \tau_c \frac{\partial}{\partial x_k} v'_i v'_k \]

\[ \beta_{ij} = \frac{1}{2} \tau_c \left( \mathbf{v'}^2 \delta_{ij} - v'_i v'_j \right) \]

are independent of \( \eta \) (in this approximation), indicating fast dynamo action.
Validity of Mean field expansion

Second order correlation approximation:
- At best marginally justified
- Works better than it should

Most general form for mean field expansion:

\[
\overline{\mathcal{E}}_i(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} d^3x' \, dt' \, K_{ij}(x, t, x', t') \, \overline{B}_j(x', t').
\]

Sufficient scale separation
- \( l_c \ll L \)
- \( \tau_c \ll \tau_L \)

leads to:

\[
\overline{\mathcal{E}} = \alpha \overline{B} + \gamma \times \overline{B} - \beta \nabla \times \overline{B} - \delta \times \nabla \times \overline{B} + \ldots
\]

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Large scale convection (M. Miesch, HAO)
Symmetry constraints

\[ \alpha, \beta, \gamma \text{ and } \delta \text{ depend on large scale symmetries of the system defining the symmetry properties of the turbulence (e.g. rotation and stratification). Additional to that the expansion} \]

\[ \mathcal{E} = \alpha \mathbf{B} + \gamma \times \mathbf{B} - \beta \nabla \times \mathbf{B} - \delta \times \nabla \times \mathbf{B} + \ldots \]

is a relation between polar and axial vectors:

- \( \mathcal{E} \): polar vector, independent from handedness of coordinate system
- \( \mathbf{B} \): axial vector, involves handedness of coordinate system in definition (curl operator, cross product)

Handedness of coordinate system pure convention (contains no physics), consistency requires:

- \( \alpha, \delta \): pseudo tensor
- \( \beta, \gamma \): true tensors
Symmetry constraints

Turbulence with rotation and stratification

- True tensors: \( \delta_{ij}, g_i, g_i g_j, \Omega_i \Omega_j, \Omega_i \varepsilon_{ijk} \)
- Pseudo tensors: \( \varepsilon_{ijk}, \Omega_i, \Omega_i g_j, g_i \varepsilon_{ijk} \)

Symmetry constraints allow only certain combinations:

\[
\alpha_{ij} = \alpha_0 (g \cdot \Omega) \delta_{ij} + \alpha_1 (g_i \Omega_j + g_j \Omega_i), \quad \gamma_i = \gamma_0 g_i + \gamma_1 \varepsilon_{ijk} g_j \Omega_k \\
\beta_{ij} = \beta_0 \delta_{ij} + \beta_1 g_i g_j + \beta_2 \Omega_i \Omega_j, \quad \delta_i = \delta_0 \Omega_i
\]

The scalars \( \alpha_0 \ldots \delta_0 \) depend on quantities of the turbulence such as rms velocity and correlation times scale.

- Isotropic turbulence: only \( \beta \)
- + Stratification: \( \beta + \gamma \)
- + Rotation: \( \beta + \delta \)
- + Stratification + Rotation: \( \alpha \) can exist
What is needed to circumvent Cowling’s theorem?

- Crucial for Cowling’s theorem: Impossibility to drive a current parallel to magnetic field
- Cowling’s theorem does not apply to mean field if a mean current can flow parallel to the mean field (since total field non-axisymmetric this is not a contradiction!)

\[
\mathbf{j} = \tilde{\sigma} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} + \gamma \times \mathbf{B} + \alpha \mathbf{B} \right)
\]

\( \tilde{\sigma} \) contains contributions from \( \eta \), \( \beta \) and \( \delta \).

Ways to circumvent Cowling:

- \( \alpha \)-effect
- anisotropic conductivity (off diagonal elements + \( \delta \)-effect)
Induction of field parallel to current (producing helical field!)

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) = \alpha \mu_0 \mathbf{j}
\]

Dynamo cycle:

\[
\mathbf{B}_t \xrightarrow{\alpha} \mathbf{B}_p \xrightarrow{\alpha} \mathbf{B}_t
\]

- Poloidal and toroidal field of similar strength
- In general stationary solutions
Dynamo cycle:

\[ \mathbf{B}_t \xrightarrow{\alpha} \mathbf{B}_p \xrightarrow{\Omega} \mathbf{B}_t \]

- Toroidal field much stronger than poloidal field
- In general traveling (along lines of constant \( \Omega \)) and periodic solutions
\[ \frac{\partial B}{\partial t} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r v_r B) + \frac{\partial}{\partial \theta} (v_\theta B) \right) = r \sin B_p \cdot \nabla \Omega \]
\[ + \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) B \]

\[ \frac{\partial A}{\partial t} + \frac{1}{r \sin \theta} v_p \cdot \nabla (r \sin \theta A) = \alpha B + \eta \left( \Delta - \frac{1}{(r \sin \theta)^2} \right) A \]

- Dimensionless measure for strength of \( \Omega \)- and \( \alpha \)-effect

\[ D_\Omega = \frac{R^2 \Delta \Omega}{\eta_t} \quad D_\alpha = \frac{R \alpha}{\eta_t} \]

- Dynamo excited if dynamo number

\[ D = D_\Omega D_\alpha > D_{crit} \]

Movie: \( \alpha \Omega \)-dynamo
Meridional flow:
- Poleward at top of convection zone
- Equatorward at bottom of convection zone

Effect of advection:
- Equatorward propagation of activity
- Correct phase relation between poloidal and toroidal field
- Circulation time scale of flow sets dynamo period

Requirement: Sufficiently low turbulent diffusivity

Movie: Flux-transport-dynamo (M. Dikpati, HAO)
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\delta \times (\nabla \times \mathbf{B})] \sim \nabla \times (\Omega \times \mathbf{j}) \sim \frac{\partial \mathbf{j}}{\partial z} \]

- similar to \( \alpha \)-effect, but additional \( z \)-derivative of current
- couples poloidal and toroidal field
- \( \delta^2 \) dynamo is not possible:

\[ \mathbf{j} \cdot \mathbf{\bar{E}} = \mathbf{j} \cdot (\delta \times \mathbf{j}) = 0 \]

- \( \delta \)-effect is controversial (not all approximations give a non-zero effect)
- in most situations \( \alpha \) dominates
Magnetic helicity (integral measure of field topology):

\[ H_m = \int \mathbf{A} \cdot \mathbf{B} \, dV \]

has following conservation law (no helicity fluxes across boundaries):

\[ \frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, dV = -2\mu_0 \eta \int \mathbf{j} \cdot \mathbf{B} \, dV \]

Decomposition into small and large scale part:

\[ \frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, dV = +2 \int \mathbf{E} \cdot \mathbf{B} \, dV - 2\mu_0 \eta \int \mathbf{j} \cdot \mathbf{B} \, dV \]

\[ \frac{d}{dt} \int \mathbf{A}' \cdot \mathbf{B}' \, dV = -2 \int \mathbf{E} \cdot \mathbf{B} \, dV - 2\mu_0 \eta \int \mathbf{j}' \cdot \mathbf{B}' \, dV \]
Dynamos and magnetic helicity

Dynamos have helical fields:
- $\alpha$-effect induces magnetic helicity of same sign on large scale
- $\alpha$-effect induces magnetic helicity of opposite sign on small scale

Saturation process (on scale $\sim R_m \tau_c$):

$$\overline{j} \cdot \overline{B}' = \frac{|B|}{|B'|} \sim \sqrt{\frac{L}{l}}$$

$$\overline{j} \cdot \overline{B}' = -\frac{\alpha \overline{B}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \overline{j} \cdot \overline{B}$$

Time scales:
- Galaxy: $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- Sun: $\sim 10^8$ years
- Earth: $\sim 10^6$ years
Non-kinematic effects

Proper way to treat them: 3D simulations
- Still very challenging
- Has been successful for geodynamo, but not for solar dynamo

Semi-analytical treatment of Lorentz-force feedback in mean field models:
- Macroscopic feedback: Change of the mean flow (differential rotation, meridional flow) through the mean Lorentz-force

\[ \bar{f} = \bar{j} \times \bar{B} + j' \times B' \]

- Mean field model including mean field representation of full MHD equations:
  Movie: Non-kinematic flux-transport dynamo
- Microscopic feedback: Change of turbulent induction effects (e.g. \( \alpha \)-quenching)
Feedback of Lorentz force on small scale motions:

- Intensity of turbulent motions significantly reduced if $\frac{1}{2\mu_0} B^2 > \frac{1}{2} \varrho v_{rms}^2$. Typical expression used

$$\alpha = \frac{\alpha_k}{1 + \frac{B^2}{B_{eq}^2}}$$

with the equipartition field strength $B_{eq} = \sqrt{\mu_0 \varrho v_{rms}}$

- Similar quenching also expected for turbulent diffusivity

- Additional quenching of $\alpha$ due to topological constraints possible (helicity conservation)

Controversial!
Symmetry of momentum and induction equation $\mathbf{v}' \leftrightarrow \mathbf{B}'$:

\[
\frac{d\mathbf{v}'}{dt} = \frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla) \mathbf{B}' + \ldots
\]

\[
\frac{d\mathbf{B}'}{dt} = (\mathbf{B} \cdot \nabla) \mathbf{v}' + \ldots
\]

\[
\mathbf{E} = \mathbf{v}' \times \mathbf{B}'
\]

Strongly motivates magnetic term for $\alpha$-effect (Pouquet et al. 1976):

\[
\alpha = \frac{1}{3} \tau_c \left( \frac{1}{\rho} \mathbf{j}' \cdot \mathbf{B}' - \mathbf{\omega}' \cdot \mathbf{v}' \right)
\]

- **Kinetic $\alpha$:** $\mathbf{B} + \mathbf{v}' \rightarrow \mathbf{B}' \rightarrow \mathbf{E}$
- **Magnetic $\alpha$:** $\mathbf{B} + \mathbf{B}' \rightarrow \mathbf{v}' \rightarrow \mathbf{E}$
From helicity conservation one expects
\[ \bar{j}' \cdot \bar{B}' \sim -\alpha \bar{B}^2 \]
leading to algebraic quenching
\[ \alpha = \frac{\alpha_k}{1 + g \frac{\bar{B}^2}{B_{eq}^2}} \]

With the asymptotic expression (steady state)
\[ \bar{j}' \cdot \bar{B}' = -\frac{\alpha \bar{B}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{j} \cdot \bar{B} \]
we get
\[ \alpha = \frac{\alpha_k + \frac{\eta_t^2}{\eta} \frac{\mu_0 \bar{j} \cdot \bar{B}}{B_{eq}^2}}{1 + \frac{\eta_t}{\eta} \frac{\bar{B}^2}{B_{eq}^2}} \]
Microscopic feedback

Catastrophic $\alpha$-quenching ($R_m \gg 1$!) in case of steady state and homogeneous $\mathbf{B}$:

$$\alpha = \frac{\alpha_k}{1 + R_m \frac{\mathbf{B}^2}{B_{eq}^2}}$$

If $\mathbf{j} \cdot \mathbf{B} \neq 0$ (dynamo generated field) and $\eta_t$ unquenched:

$$\alpha \approx \eta_t \mu_0 \frac{\mathbf{j} \cdot \mathbf{B}}{\mathcal{B}^2} \sim \frac{\eta_t}{L} \sim \frac{\eta_t}{l_c} \frac{l_c}{L} \sim \alpha_k \frac{l_c}{L}$$

- In general $\alpha$-quenching dynamic process: linked to time evolution of helicity
- Boundary conditions matter: Loss of small scale current helicity can alleviate catastrophic quenching
- Catastrophic $\alpha$-quenching turns large scale dynamo into slow dynamo
Stationary state reached on time scale $R_m \tau_c$:
- Galaxy: $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- Sun: $\sim 10^8$ years
- Earth: $\sim 10^6$ years
- Universe too young for galaxies to worry about stationary state!
- Sun, geodynamo had enough time too saturate
  - Sun: Possibility that helicity loss through photosphere alleviates quenching
  - Geodynamo: $R_m \sim 300$ not that catastrophic?
- No observational evidence for catastrophic $\alpha$-quenching, but fundamental question for theory!
3D simulations

Why not just solving the full system to account for all non-linear effects?

- Most systems have $R_e \gg R_m \gg 1$, requiring high resolution.
- Large scale dynamos evolve on time scales $\tau_c \ll t \ll \tau_\eta$, requiring long runs compared to convective turn over.
- 3D simulations successful for geodynamo:
  - $R_m \sim 300$: all relevant magnetic scales resolvable.
  - Incompressible system.
- Solar dynamo: Ingredients can be simulated:
  - Compressible system: density changes by $10^6$ through convection zone.
  - Boundary layer effects: Tachocline, difficult to simulate (strongly subadiabatic stratification, large time scales).
  - Magnetic structures down to 1000 km most likely important.
    Evolve $5000^3$ box over 1000 $\tau_c$!
  - Small scale dynamos can be simulated (for $P_m \sim 1$).
Summarizing remarks

Destruction of magnetic field:

- Turbulent diffusivity: cascade of magnetic energy from large scale to dissipation scale (advection + reconnection)
- Enhances dissipation of large field by a factor $R_m$

Creation of magnetic field:

- Small scale dynamo (non-helical)
  - Amplification of field on and below scale of turbulence
  - Stretch-twist-fold-(reconnect)
  - Produces non-helical field and does not require helical motions
  - **Current research**: behavior for $P_m \ll 1$

- Large scale dynamo (helical)
  - Amplification of field on scales larger than scale of turbulence
  - Produces helical field and does require helical motions
  - Requires rotation + additional symmetry direction
  - (controversial $\Omega \times J$ effect does not require helical motions)
  - **Current research**: catastrophic vs. non-catastrophic quenching