Laboratory Exercise
Solar Convection and Dynamo

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1 Introduction

1.1 The Inner Turmoil of Stars

It's not easy being a star. Whether you're a Hollywood celebrity or a luminescent sphere of plasma, you face some extraordinary challenges. In order to shine in the latter case, you must find a way to to transport energy from your core where it is liberated by thermonuclear fusion to your surface where it can be radiated into space.

In the core of the Sun, the outward diffusion of photons provides a sufficient means of energy transport. However, in the relatively cool solar envelope the plasma is more opaque, forming a bottleneck to radiative heat transport. Thermal gradients build up until the star's only recourse is to bodily move hot plasma upward and cool plasma downward in the familiar process of thermal convection.

Nearly all stars rely on convective energy transport across at least some portion of their interior. When stars convect, they do so vigorously; stellar convection is highly turbulent. This may be quantified by the Reynolds number, R_e, which measures the magnitude of nonlinear advection relative to viscous diffusion. In stellar convection zones, $R_e = UL/\nu > 10^{12}$, where U and L are characteristic velocity and length scales and ν is the kinematic molecular viscosity.

Turbulent flow in an electrically conducting fluid inevitably generates magnetic fields. This is true provided that the flow is three-dimensional (see §4) and the magnetic Reynolds number $R_m = UL/\eta$, where η is the (molecular) magnetic diffusivity, is sufficiently large. This is easily satisfied in stars, where $R_m \sim 10^5$–10^{10}. Thus convection, together with the rotational shear and meridional circulation it generates, is ultimately responsible for the rich display of magnetic activity so evident in the Sun and other stars and so central to the discipline of Heliophysics.

1.2 A Simple Model

The simplest system that exhibits thermal convection is a two-dimensional (2D) Cartesian layer heated from below and cooled from above. Here we label the vertical dimension z and the horizontal dimension y. We assume that the layer has a finite vertical extent D and an infinite (periodic) horizontal extent. For simplicity we adopt the Boussinesq approximation whereby we assume that D is much smaller than the density, pressure, and temperature scale heights of the background stratification, that flow speeds are much less the sound speed, and that pressure variations produced by the convection are relatively small such that buoyancy-inducing density variations are inversely proportional to temperature variations[1].

The relevant non-dimensional Boussinesq, 2D equations are as follows:

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad , \tag{1}$$

$$\frac{\partial\omega}{\partial t} = -\boldsymbol{\nabla}\cdot(\omega\mathbf{v} - J\boldsymbol{B}) + R_a P_r \frac{\partial T}{\partial y} + P_r \nabla^2\omega \quad , \tag{2}$$

$$\frac{\partial T}{\partial t} = -\boldsymbol{\nabla}\cdot(T\mathbf{v}) + \nabla^2 T \quad , \tag{3}$$

$$\frac{\partial A}{\partial t} = -\boldsymbol{\nabla}\cdot(A\mathbf{v}) + R_m^{-1} \nabla^2 A \quad . \tag{4}$$

Here $\mathbf{v}$ and $\boldsymbol{B}$ represent the velocity and magnetic fields and T is the temperature variation relative to a hydrostatic background stratification. The fluid vorticity ω and the current density J are given by

$$\omega = [\boldsymbol{\nabla}\times\mathbf{v}]\cdot\hat{\boldsymbol{x}} \tag{5}$$

$$J = [\boldsymbol{\nabla}\times\boldsymbol{B}]\cdot\hat{\boldsymbol{x}} = -\boldsymbol{\nabla}^2 A \tag{6}$$

and the scalar magnetic potential A is defined such that

$$\boldsymbol{B} = \boldsymbol{\nabla}\times(A\hat{\boldsymbol{x}}) \quad . \tag{7}$$

The equations are made non-dimensional using the length scale D and a thermal diffusion time scale D^2/κ where κ is the thermal diffusivity. In sections 2 and 3 we focus on the non-magnetic case $A = \boldsymbol{B} = 0$. Here there are two nondimensional parameters, the first of which is the Rayliegh number R_a, which quantifies the magnitude of the buoyancy force relative to viscous and thermal dissipation. The second parameter is the Prandtl number $P_r = \nu/\kappa$, which quantifies the relative efficiency of viscous and thermal diffusion. In §4 we briefly consider magnetism, in which case there is a third parameter, the magnetic Reynolds number R_m. For this choice of nondimensional scaling, $R_m \equiv UL/\eta = P_m/P_r$ where $P_m = \nu/\eta$ is the magnetic Prandtl number.

The IDL program *convection.pro* solves equations (1)–(4) under the assumption of isothermal, impenetrable, stress-free, perfectly conducting boundaries such that

$$\omega = \frac{\partial^2\omega}{\partial z^2} = A = 0 \qquad (z = 0, 1) \tag{8}$$

and

$$T = 0.5 \quad (z = 0) \qquad\qquad T = -0.5 \quad (z = 1). \tag{9}$$

[1]The Boussinesq approximation is discussed in many introductory fluid dynamics textbooks; see, for example Chandrasekhar (1961) or Kundu (1990). For a thorough exposition in a magnetohydrodynamic (MHD) context see Proctor & Weiss (1982)

2 Convective Instability

An equilibrium solution to equations (1)–(4) is given by

$$\omega = A = 0 \tag{10}$$

and

$$T = T_e \equiv \frac{1}{2} - z \quad . \tag{11}$$

We can check the linear stability of this equilibrium by introducing temperature perturbations of the form

$$T = T_e + \theta \tag{12}$$

and then expressing θ in terms of sinusoidal basis functions such that

$$\theta(y, z, t) = \Theta_0 \sin(n\pi z) \exp(\imath ky + \sigma t) \quad , \tag{13}$$

with a similar expansion for ω. If $A = 0$ initially, equation (4) ensures that it will remain zero so we need not bother with magnetism in this section[2]. The result is that the equilibrium is unstable (perturbations grow exponentially in time) if the Rayleigh number exceeds a critical value given by (Chandrasekhar, 1961)

$$\text{unstable if } R_a > R_k = \frac{(n^2\pi^2 + k^2)^3}{k^2} \quad . \tag{14}$$

In our bounded, periodic system, the horizontal wavnumber is given by $k = 2\pi m/L_y$ where L_y is the aspect ratio of the computational domain (y ranges from 0 to L_y). In this laboratory exercise we'll focus on $L_y = 2$, although you are free to explore different aspect ratios if time permits (see §2, Step 5 below).

Each mode will have a different value of R_k but the overall stability of the layer will be governed by the mode with the lowest value. For $L_y = 2$, the critical mode turns out to be the gravest mode, that with $n = m = 1$. Substituting these values into equation (15) yields a critical Rayleigh number of

$$R_c = \min(R_k) = 779 \qquad\qquad (L_y = 2) \quad . \tag{15}$$

Laboratory Exercise

Step 1: Start IDL. At the IDL command line type

```
IDL> convection
```

Congratulations! You have just become a modeler; you are now running a convection simulation! On the screen you will see the various parameters that describe the simulation, set to their default values. Note how many there are just for this simple numerical model! You can reset any of these parameters on the command line as described below. However, keep in mind that the underlying (non-magnetic) physical/mathematical model described

[2]The linear stability problem becomes much richer when $A \neq 0$; for a taste see Proctor & Weiss (1982).

by equations (1)-(3) only has two parameters, R_a and P_r. Note their values now and note also the value of R_k, the critical Rayleigh number corresponding specifically to the initial perturbation. The quantity R_k is not a parameter of the problem that you can specify; rather, it is computed from equation (14) given the horizonal and vertical wavenumbers of the initial perturbation, $k = 2\pi m/L_y$ and $n\pi$. In the numerical code, m and n are set by the parameters nky and nkz.

Wait for the IDL command line prompt to return. That means the simulation is finished. We said above [see eq. (15)] that the critical Rayleigh number R_c is equal to 779 (recall that $R_k \geq R_c$). Let's start with a simulation with a value of R_a less than R_c. Check the output on your screen (you may want to make the window bigger so you can see it all). Is the value of R_a listed less than 779? If so, proceed to Step 2. If not, you can run another simulation with a different Rayleigh number by typing, e.g.

```
IDL> convection, R_a = 700
```

Note that IDL is case insensitive so `r_a` is the same as `R_a`. Wait for the IDL prompt to return and then proceed to Step 2.

Step 2: Do you expect this system to be stable or unstable? In order to check, type:

```
IDL> convection_scalar
```

This plots the value of the volume-integrated kinetic energy $E_k = \int v^2 dA$ (solid line) and the integrated temperature variance $\int T^2 dA$ (dotted line) as a function of time on the upper plot. The lower plot shows the kinetic energy growth rate as a function of time (solid line), defined as $\gamma = d \ln E_k/dt$. Horizontal lines indicate exponential behavior. The dot-dashed line indicates zero growth rate (statistically steady) for comparison. You can specify one or both of the vertical axis ranges by typing, for example,

```
IDL> convection_scalar,yr1=[1.e-20,1.0],yr2=[-4,2]
```

The kinetic energy of the initial perturbation should exponentially grow or decay depending on whether the system is unstable or stable. This is the exponential growth/decay phase when $\gamma = 2$ times[3] the real part of σ as expressed in equation (13). Note the value of R_a and the approximate value of γ during the exponential growth/decay phase as read off the plot. Write them down - you may want them later in Step 5.

Depending on your value of R_a, you may notice that the rate of decay flattens out at long times, after the kinetic energy drops below about 10^{-16}. This number may ring a bell; double-precision numbers in computer codes generally only have 16 significant digits. Thus, you might guess that this "flattening out" is not "real", in the sense that it is not an accurate solution of the governing equations (1)-(3). In other words, it is *numerical noise*. You'd be right; if it were an exact solution it would continue to decay exponentially at the same rate forever. This gives you a taste of some of the limitations of numerical modeling; always keep them in mind when interpreting a numerical solution of a mathematical system!

Step 3: Now try a value of R_a greater than R_c, say

```
IDL> convection,R_a = 2000
```

[3]The factor of two comes in because we're plotting the kinetic energy, which goes as the velocity squared.

wait for the prompt to come back, and then look at the energy traces

```
IDL> convection_scalar
```

Is this one stable or unstable? Write down the value of R_a and the value of γ during the exponential growth phase. Does exponential growth persist indefinitely, or does the slope flatten out? In Step 2, we said the flattening out in that simulation was not real. Is this one real? Why would the exponential growth stop? Note the saturation value of the kinetic energy (where E_k approaches a constant value, at least in a time-averaged sense).

What happens to the temperature variance after the kinetic energy saturates? You can check this on a linear scale by typing

```
IDL> convection_scalar,/temperature
```

Now the top plot shows just the temperature variance on a linear (as opposed to logarithmic) scale. The bottom plot is the kinetic energy growth rate as before, approching zero at saturation. Any idea why the temperature variance does what it does? If not, stay tuned - you'll get a better feel for this as we proceed (see §3).

Step 4: In order to see what the convection looks like, type

```
IDL> convection_movie
```

This displays the 2D solution on the screen and writes the images to disk where they can be combined as movie frames using a utility such as the Apple QuickTime player. The default is to show the temperature (q=3), with red denoting hot fluid and blue cold. However, you can also type:

```
IDL> convection_movie,q = 1
```

to see the vorticity ω. Red denotes counter-clockwise circulations and blue denotes clockwise. Other options are written on your screen (or look in the file *convection_movie.pro*).

Step 5: Play around! As time permits, explore a few other parameter sets. You may wish to do sections 3 and 4 first and come back to this if you have time. Questions to explore include: Does the growth rate γ depend on R_a or P_r? What about the saturation level of the kinetic energy, which is related to the Reynolds number[4]. Does R_c depend on P_r or L_y? What happens if the initial perturbation has a different wavenumber ($n > 1$ and/or $m > 1$)? What happens if the value of R_a is less than the critical value for that particular input perturbation, R_k, but greater than the ultimate critical value R_c? What happens if you pick an R_a that is very large or very small? Note that you can combine parameters, for example:

```
IDL> convection, R_a = 5000, P_r = 2, Ly = 4
```

Implications:
Clearly, in order for our layer to be convectively unstable, the temperature must decrease

[4]In our nondimensional units, the Reynolds number can be defined as $R_e = \sqrt{E_k}/P_r$, where E_k is the mean kinetic energy.

outward, $\partial T/\partial z < 0$. In a compressible fluid the corresponding criterion is the Schwarzchild criterion, $\partial S/\partial z < 0$, where S is the specific entropy. However, this is not enough. In order to convect, the buoyancy force must overcome the stabilizing effects of viscous and thermal diffusion. This is the physics behind the critical Rayliegh number, R_c. Although stars are comfortably supercritical ($R_a > 10^{20}$), numerical simulations are less self-assured ($R_a \sim 10^5$–10^7).

3 Heat Transport and Boundary Layers

As emphasized in §1.1, convection is, above all else, a means to transport heat. We can make this explicit by writing the thermal energy equation (3) as follows:

$$\frac{\partial T}{\partial t} = -\boldsymbol{\nabla}\cdot(T\mathbf{v} - \boldsymbol{\nabla}T) = -\boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{F}}_e + \boldsymbol{\mathcal{F}}_d) \quad , \tag{16}$$

where $\boldsymbol{\mathcal{F}}_e = T\mathbf{v}$ is the convective enthalpy flux and $\boldsymbol{\mathcal{F}}_d = -\boldsymbol{\nabla}T$ is the heat flux due to thermal diffusion. We're most interested in the vertical transport, so we can average equation (16) over y to obtain

$$\frac{\partial \langle T \rangle}{\partial t} = -\frac{\partial}{\partial z}(F_e + F_d) \quad , \tag{17}$$

where $F_e = \langle T v_z \rangle$, $F_d = -\langle \partial T/\partial z \rangle$, and angular brackets denote averages over y.

We now define the Nusselt number N_u as the heat flux relative to F_{nc}, which is the heat flux that would have prevailed if there were no convection.

$$N_u = \frac{F_e + F_d}{F_{nc}} \quad . \tag{18}$$

Without convection, the transport would be purely diffusive and T would be given by equation (11), so $F_{nc} = -\partial T/\partial z = 1$. Now, if the system is in a statistically steady state, the time-averaged value of N_u will be independent of height, z. Energy will enter through the bottom boundary, pass through the layer, and exit through the top boundary. Thus, the value of N_u at the boundaries will tell us the energy flux through the layer. Since there is no flow through the top and bottom boundaries, then $F_e = 0$ at those boundaries. So, the heat flux through the boundaries is from diffusion alone. In fact, in terms of our non-dimensional variables, the heat flux through the top and bottom boundaries is just the vertical temperature gradient at those boundaries, so the Nusselt number is just;

$$N_u = -\left.\frac{\partial \langle T \rangle}{\partial z}\right|_b \tag{19}$$

evaluated at either the top or the bottom boundary.

Laboratory Exercise

Step 1: Run a simulation with a mildly supercritical Rayliegh number of 2000 by entering the following in IDL:

```
IDL> convection, R_a = 2000
```

Wait for the prompt, then take a look at the vorticity structure by typing

```
IDL> convection_movie,q = 1
```

You should see a few big convective rolls.

Step 2: Now check out the temperature structure as follows:

```
IDL> convection_movie,q = 3
```

Note how upflows and downflows have pulled warm and cool fluid from the boundaries into the interior of the layer. To see how this has changed the mean temperature stratification type

```
IDL> Tmean_movie
```

The solid line shows the mean temperature $\langle T \rangle$ and the dotted line shows the initial equilibrium profile T_e as in equation (11). The animation shows how $\langle T \rangle$ changes with time. The value of the Nusselt number at different times is printed to the screen and the final value is noted on the plot, evaluated as an average of the vertical temperature gradient at the upper and lower boundaries according to equation (19).

What does this imply about the efficiency of convection? In order to answer this, realize that a value of $N_u = 2$, say, means that twice as much heat is passing through the layer as would have been the case without convection. Note also the nature of the mean temperature profile. By pushing most of the temperature gradient to the boundaries, convection has made the middle of the layer almost isothermal (constant temperature).

Step 3: What do you think will happen if we increase the Rayleigh number? Higher values of R_a require higher spatial resolution in order to adequately capture the structure of the boundary layers and the turbulent flows that arise. High-resolution simulations take too long to run in the space of this laboratory exercise but we've made some results from a higher resolution case available in the directory "highres". This is for a simulation with $R_a = 10^6$ and $P_r = 5$.

To see what the temperature structure of the convection looks like, go to the highres directory and view the quicktime movie temperature.mov. How is the structure different from the lower-Rayleigh number case? Note in particular the thin thermal plumes sprouting from the boundary layers. Now check out the vorticity structure in vorticity.mov. Now watch the evolution of the mean temperature for the high resolution case in Tmean.mov. What has happened with the width of the boundary layers and the value of N_u?

If the convection were three-dimensional, it would be turbulent in this parameter regime, with plumes continually sprouting from the boundary layers and dissipating. However, 2D turbulence exhibits self-organization processes that favor an ultimate state with two big rolls. Still, in both 2D and 3D, if you were to keep increasing R_a you'd find that the boundary layers would get thinner and thinner and Nu would get bigger and bigger.

Implications:

Is it any wonder that stars employ convection as a means to transport heat outward from their interiors? High values of N_u imply efficient heat transport, and the higher the R_a, the higher the N_u. Great effort has been devoted to investigating how N_u and R_e scale with R_a and P_r in laboratory experiments, numerical simulations, and theoretical models, particularly in the highly turbulent regime $R_a >> R_c$. For an overview, see the recent review article by Ahlers *et al.* (2009). Much of this work focuses on the thermal boundary layers that mediate the heat exchange between the the boundaries and the fluid and thereby regulate the buoyancy driving. Thus, boundary layers play an essential dynamical role regardless of how thin they may be.

4 Convection and Magnetism

One of our main motivations for investigating solar and stellar convection is the essential role it plays in generating the magnetic fields that lie at the root of solar and stellar variability. Can the convective flows we've been studying in this laboratory exercise generate magnetic fields? This is the question we'll be concerned with in this brief but important section.

Laboratory Exercise

Step 1: Let's begin with the same mildly supercritical case that we began with in §3. If you haven't run anything since, then no action is needed but it doesn't hurt to run it again just to make sure you have the right data:

```
IDL> convection, R_a = 2000
```

To continue this simulation with the addition of a small seed magnetic field type

```
IDL> convection, R_a = 2000, restart = 2000, /magnetism
```

Step 2: Wait for the prompt, then check the evolution of the magnetic energy by typing

```
IDL> convection_scalar, sim = 3
```

This is the same output as in previous sections but now the magnetic energy $E_m(t) = \int B^2 dA$ is plotted as a dashed line in the upper plot and the lower plot shows the magnetic growth rate $d \ln E_m / dt$. Positive values imply growth, negative values imply decay.

Step 3: Take a look at the structure of the magnetic field by typing

```
IDL> convection_movie, sim = 3, q = 4,/noscale
```

The `/noscale` bit means that the color table in each image is scaled to it own minimum and maximum values so you can see what the field looks like as its amplitude changes (omitting it sets a scale based on the final state). The q=4 selects the magnetic potential A for plotting, so contour lines are equivalent to field lines.

Step 4: You should have found in Step 3 that the magnetic energy grows initially, as the convection sqeezes and stretches out magnetic fields. However, diffusion ultimately catches up and dissipates the field. Might it be that our magnetic Reynolds number is just too low (§1.1)? You may try different parameter values if you wish (recall that in our system

$R_m = P_m/P_r$ so you can change it by changing P_m) but you'll find that the answer is always the same; no matter how big R_m is, the magnetic energy will eventually diffuse away. Always. So what's going on?

Implications:
Convection breeds magnetism, but not in two dimensions. Cowling's anti-dynamo theorem says that a strictly 2D dynamo is not possible. Field lines may get stretched and amplified in one direction but they'll get tangled and squeezed in the perpendicular direction such that ohmic diffusion ultimately wins out. There are many fascinating aspects of 2D magnetoconvection (see Proctor & Weiss, 1982) but dynamo action is not one of them.

Two-dimensional dynamo models do exist, but in order to be viable, they must include parameterizations for three-dimensional processes that are not captured explicitly in the simulation. The most familiar example is the α-effect of mean-field dynamo theory.

References

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