Kinetic Theory

Amitava Bhattacharjee
Princeton Plasma Physics Laboratory, Princeton University

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Goal of this lecture

• Review a few basic plasma concepts in plasma kinetic theory that underlie the lectures later in the week.

• There are several excellent text books: Nicholson (out of print), Goldston and Rutherford, Boyd and Sanderson.

• The book I am most familiar with is by Gurnett and Bhattacharjee, from which most of the material is taken.
Plasma: levels of description

Plasma is an ensemble of charged particles, capable of exhibiting collective interactions.

Levels of Description:

• Single-particle dynamics in *prescribed* electric and magnetic fields
• Plasmas as fluids in 3D configuration space moving under the influence of *self-consistent* electric and magnetic fields
• Plasmas as kinetic fluids in 6D $\mu$-space (that is, configuration and velocity space), coupled to *self-consistent* Maxwell’s equations.
Single-Particle Orbit Theory

Newton’s law of motion for charged particles

\[ m \frac{dv}{dt} = q(E + v \times B) \]

Guiding-Center: A very useful concept
Single-Particle Orbit Theory

ExB Drift

\[ m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

Consider \( \mathbf{E} = \text{const.}, \mathbf{B} = \text{const.} \).

The charged particles experience a drift velocity, perpendicular to both \( \mathbf{E} \) and \( \mathbf{B} \), and independent of their charge and mass.

\[ \mathbf{V}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \]
\[ \vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \]
Gradient B drift

\[ V_G = \frac{w_{\perp}}{qB} \left( \frac{\mathbf{B} \times \nabla \mathbf{B}}{B^2} \right), \]

\[ w_{\perp} = \frac{1}{2} \omega_c^2 \rho_c^2 \]
Curvature drift

\[ V_C = \frac{2w_||}{qB^2} \left( \frac{R_C \times B}{R_C^2} \right) \]

\[ w_|| = \frac{1}{2} m v_||^2 \]
The Ring Current in Earth’s Magnetosphere: An Example
Kinetic Description of Plasmas

Distribution function \( f(\mathbf{r}, \mathbf{v}, t) \)

Total number of particles \( N = \iiint d\mathbf{x} d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) \) 

phase space

Example: Maxwell distribution function

\[
f = n_0 \exp \left( -\frac{mv^2}{2kT} \right), \quad n_0 = \frac{N}{V}
\]
Boltzmann-Vlasov Equation

Motion of an incompressible phase fluid in $\mu$–space (6D)

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{v}} + \frac{q}{m} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \quad s = e, i$$

In the presence of collisions

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{v}} + \frac{q}{m} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left( \frac{\partial f_s}{\partial t} \right)_c$$
Properties of the Vlasov Equation

1. The Vlasov equation conserves the total number of particles $N$ of a species, which can be proven, for the one-dimensional case, as follows:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \iint f \, dx \, dv = - \iint v \frac{\partial f}{\partial x} \, dx \, dv - \iint a \frac{\partial f}{\partial v} \, dx \, dv$$

2. Any function, $g\left[\frac{1}{2}mv^2 + q \Phi(x)\right]$, which can be written in terms of the total energy of the particle, is a solution of the Vlasov equation.

3. The Vlasov equation has the property that the phase-space density $f$ is constant along the trajectory of a test particle that moves in the electromagnetic fields $E$ and $B$. Let $[x(t), v(t)]$ be the trajectory that follows from the equation of motion $m\ddot{v} = q(E + v \times B)$ and $\dot{x} = v$, then

$$\frac{df(x(t), v(t), t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot v + \frac{\partial f}{\partial v} \cdot \frac{q}{m} (E + v \times B) = 0.$$
Properties of the Vlasov Equation

4. The Vlasov equation is invariant under time reversal, \((t \to -t), (v \to -v)\). This means that there is no change in entropy for a Vlasov system.

\[
\frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial t} + \frac{q}{m} \left( E + v \times B \right) \cdot \frac{\partial f_s}{\partial x} = \left( \frac{\partial f_s}{\partial t} \right)_c
\]

Contrast with Boltzmann’s equation

Boltzmann’s \(H\) functional

\[
S(f) = -H(f) := -\int_{\Omega_x \times \mathbb{R}^3} f(x, v) \log f(x, v) \, dv \, dx
\]

Boltzmann identifies \(S\) with the entropy of the gas and proves that \(S\) can only increase in time (strictly unless the gas is in a hydrodynamical state) — an instance of the Second Law of Thermodynamics.
Boltzmann’s H-Theorem

If this is true, then \( f \) becomes Gaussian as \( t \to \infty \! \)
Vlasov-Poisson equations: requirements of self-consistency in an electrostatic plasma

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \Phi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0
\]

\[
\mathbf{E} = -\nabla \Phi
\]

\[
\nabla \cdot \mathbf{E} = -\nabla^2 \Phi = 4\pi \rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s
\]
Linear Plasma Waves

\[ f_e(x, v, t) = f_{e0}(v) + f_{e1}(x, v, t) \]

\[ f_{e0}(v) = n_{e0} \left( \frac{m_e}{2\pi k_B T_e} \right)^{1/2} \exp \left\{ -\frac{m_e v^2}{2k_B T_e} \right\} \]

\[ f_{e1} = \hat{f}_{e1} \exp[i(kx - \omega t)]. \]

Linearizing the Vlasov equation, and using the wave representation, we obtain

\[ \frac{\partial f_{e1}}{\partial t} + v \frac{\partial f_{e1}}{\partial x} - \frac{e}{m_e} E_1 \frac{\partial f_{e0}}{\partial v} = 0 \]

\[ -i\omega \hat{f}_{e1} + ik v \hat{f}_{e1} - \frac{e}{m_e} \hat{E}_1 \frac{\partial f_{e0}}{\partial v} = 0, \]
Linear Plasma Waves

\[ f_e(x, v, t) = f_{e0}(v) + f_{e1}(x, v, t) \]

\[ f_{e0}(v) = n_{e0} \left( \frac{m_e}{2\pi k_B T_e} \right)^{1/2} \exp \left\{ -\frac{m_e v^2}{2k_B T_e} \right\} \]

\[ f_{e1} = \hat{f}_{e1} \exp[i(kx - \omega t)]. \]

\[ \hat{f}_{e1} = i \frac{e}{m_e} \frac{\partial f_{e0}}{\partial v} \hat{E}_1. \]

The vanishing of the denominator \((\omega - kv)\) causes a singularity in the perturbed distribution function, which we will have to address carefully. The electrons with \(v=\omega/k\) are called resonant particles.
Linear Dispersion Relation

\[ D(k, \omega) = 1 - \sum_x \frac{\omega_{p_x}^2}{k^2} \int_L \frac{\partial F_{x0}/\partial u}{u - \omega/|k|} \, du = 0 \]

then gives for the dispersion equation for weakly damped electrostatic waves in a field-free plasma

\[ 1 - \sum_x \frac{\omega_{p_x}^2}{k^2} \left(1 + i\omega_i \frac{\partial}{\partial \omega_i}\right) \oint \frac{\partial F_{x0}(u)/\partial u}{u - \omega_r/|k|} \, du \]

\[ + \pi i \left[ \frac{\partial F_{x0}(u)}{\partial u} \right]_{u=\omega_r/|k|} = 0 \quad \text{(8.5.9)} \]
Distribution function and Landau damping

\[ \omega_l = \pm \omega_{pe} \left(1 + \frac{3}{2} k^2 \lambda_D^2\right) + i\gamma_l(k) \]

\[ \gamma_l(k) = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{pe}}{k^3 \lambda_D^3} \exp \left(-\frac{1}{2k^2 \lambda_D^2} - \frac{3}{2}\right) \]
Non-Maxwellian Distributions

Figure 5. Typical ion energy spectrum in Saturn’s magnetosphere measured by the LECP instrument on Voyager 2 at $10R_S$ (from Krimigis, 1982).
COLLISIONLESS DAMPING OF ELECTROSTATIC PLASMA WAVES*

J. H. Malmberg and C. B. Wharton
John Jay Hopkins Laboratory for Pure and Applied Science,
General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 6 July 1964)

DISPERSION OF ELECTRON PLASMA WAVES*

J. H. Malmberg and C. B. Wharton
John Jay Hopkins Laboratory for Pure and Applied Science,
General Atomic Division of General Dynamics Corporation, San Diego, California
(Received 31 May 1966)
Landau Damping: The Measurement

Important key observation...

FIG. 3. Logarithm of damping length vs phase velocity squared. The solid curve is theory of Landau for a Maxwellian distribution with a temperature of 10.5 eV.
Quasilinear theory: application to scattering due to wave-particle interactions

• Consider electrostatic Vlasov equation

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial t} - \frac{q}{m} \nabla \Phi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0. 
\]

Split every dependent variable into a mean and a fluctuation

\[
f_s = \langle f_s \rangle + f_{s1}, \quad \langle f_{s1} \rangle = 0
\]
Quasilinear Diffusion

It follows after some algebra that the mean or average distribution function obeys a diffusion equation:

$$\frac{\partial}{\partial t} \langle f_s \rangle = \frac{\partial}{\partial \mathbf{v}} \left( \mathbf{D} \cdot \frac{\partial}{\partial \mathbf{v}} \langle f_s \rangle \right)$$

Here $\mathbf{D}$ is a diffusion tensor, dependent on wave fluctuations. These fluctuations can be a proxy for collisions as far as the average distribution function is concerned.
Fluid Models

A primary fluid model of focus in this summer school is Magnetohydrodynamics (MHD). It treats the plasma as a single fluid, without distinguishing between electrons or protons, moving under the influence of self-consistent electric and magnetic fields.

It can be derived from kinetic theory by taking moments (integrating over velocity space), and making some drastic approximations.
Fluid equation of continuity

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0
\]

\[
0 = \frac{\partial}{\partial t} \int f \, dv + \frac{\partial}{\partial x} \int \mathbf{v} f \, dv + a[f]_{-\infty}^{\infty} = \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu)
\]
Fluid momentum equation

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0
\]

\[
0 = \frac{\partial}{\partial t} \int m \rho f \, dv + \frac{\partial}{\partial x} \left( \int v^2 f \, dv \right) + a \int v \frac{\partial f}{\partial v} \, dv \times m
\]

\[
= \frac{\partial}{\partial t} \int m \rho f \, dv + \frac{\partial}{\partial x} \left[ \int m (v - u)^2 f \, dv + n \mu u^2 \right]
\]

\[
+ a \left( \left[ v f \right]_{-\infty}^{\infty} - \int f \frac{dv}{dv} \, dv \right) \times m
\]

\[
= \frac{\partial}{\partial t} (n \mu u) + \frac{\partial p}{\partial x} + u \frac{\partial}{\partial t} (n \mu u) + (n \mu) \frac{\partial u}{\partial x} - n \mu a
\]

\[
= n m \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} - n \mu a,
\]

\[
p = \int m (v - u)^2 f \, dv
\]