How to get anywhere in the Solar System on a single tank of fuel

A dilettante dabbles in orbital mechanics

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Gravity Assists:

Why?

How?

Why so many?

Parker Solar Probe (PSP)

Voyager 1 & 2

Cassini
From Fran Bagenal’s lecture yesterday:

*Sample Trajectory to Uranus:*
*Earth-Venus-Earth-Earth-Jupiter*
*Gravity Assist*

... say what!?
eccentricity: \( e = \frac{r_a - r_p}{r_a + r_p} \)

\( r_a = (1+e)a \quad r_p = (1-e)a \)

S/C orbits Sun ballistically:

ellipse w/ Sun @ focus

(Kepler’s 1st)
eccentricity: \[ e = \frac{r_a - r_p}{r_a + r_p} \]

\[ r_a = (1 + e) a \quad r_p = (1 - e) a \]

\[ E = \frac{1}{2} \mathbf{v}^2 - \frac{GM}{r} = -\frac{GM}{2a} \]

\[ \frac{a}{R} = \frac{1}{2 - \mathbf{v}^2 / \mathbf{v}_o^2} \]

\[ p_\phi = r^2 \mathbf{v}_\phi = GM a (1 - e^2) \]

\[ e^2 = \frac{R \mathbf{v}_\phi^2}{a \mathbf{v}_o^2} - 1 \]

\[ \mathbf{v}_o = \sqrt{\frac{GM}{R}} \]
\[
a = \frac{1}{R \left(2 - \frac{v^2}{v_o^2}\right)}
\]

\[
e^2 = \frac{R \frac{v^2}{\phi}}{a \frac{v_o^2}{v_o}} - 1
\]

\[
r_a = (1 + e) a \quad r_p = (1 - e) a
\]
$r_p = 0.04 \, R$

$r_a = 1.04 \, R$
$r_p = 0.04 \, R$

$r_a = R$

grazing orbits

$r = R$

$v_r = 0$
$r_a = 3R$

$r_p = R$

$\dot{r}_r = 0$

grazing orbits

perihelion$/R$

$a_{\text{helion}}/R$

$r = R$

$v = 0$

$v_2$
\[
\frac{r_a}{R} = \frac{1 + e}{2 - v^2 / v_0^2} \quad v_r/v_o
\]

\[
v = \sqrt{2} \cdot v_0
\]

\[
r_a \rightarrow \infty
\]
Match the mission to the orbit

☐ MAVEN:
   Earth → Mars direct

☐ Helios 2:
   1976: sampled solar wind 0.29 < r < 1 AU

☐ New Horizons:
   Earth → Jupiter → out passing Pluto & Ultima Thule

☐ Cassini:
   Earth → Venus → Earth → Venus → Saturn
MAVEN: Earth $\rightarrow$ Mars direct

Helios 2:
1976: sampled solar wind $0.29 < r < 1$ AU

New Horizons:
Earth $\rightarrow$ Jupiter $\rightarrow$ out passing Pluto & Ultima Thule

Cassini:
Earth $\rightarrow$ Venus $\rightarrow$ Earth $\rightarrow$ Venus $\rightarrow$ Saturn
MAVEN:
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R=1 AU
MAVEN: Earth $\rightarrow$ Mars direct

Helios 2: 1976: sampled solar wind $0.29 < r < 1$ AU

New Horizons: Earth $\rightarrow$ Jupiter $\rightarrow$ out passing Pluto & Ultima Thule

Cassini: Earth $\rightarrow$ Venus $\rightarrow$ Earth $\rightarrow$ Venus $\rightarrow$ Saturn

VEN: $\rightarrow$ Mars direct

Venus 1 Flyby 26 Apr 1998

Venus 2 Flyby 24 Jun 1999

Venus Targeting Maneuver 3 Dec 1998

Earth Flyby 18 Aug 1999

Saturn Orbit Insertion 1 Jul 2004

0.29 < $r$ < 1 AU

Horizons: $\rightarrow$ Jupiter $\rightarrow$ out passing Pluto & Ultima Thule
Relative to Earth:
\[ \Delta v_e = v - v_{o,e} \phi \]
\[ |\Delta v_e| = 12.7 \text{ km/s} \]
\[ m \frac{dv}{dt} = F = |\dot{m}|u \]

\[ \Delta v = u \ln \left( \frac{\left| \Delta m_f \right| + m_p}{m_p} \right) \approx u \ln \left( \frac{\left| \Delta m_f \right|}{m_p} \right) \]

**Impulse:** \[ \int F \, dt = \int u \, \dot{m} \, dt = u \, |\Delta m| \]

**Specific impulse:** \[ u = \frac{\int F \, dt}{|\Delta m|} \quad \text{[m/s] momentum/mass} \]
\[ m \frac{dv}{dt} = F = |\dot{m}| u \]

\[ \Delta v \approx u \ln \left( \frac{\Delta m_f}{m_p} \right) \]

Specific impulse depends on fuel:

\[ 2\text{H}_2 + \text{O}_2 \rightarrow 2 \times (\text{H}_2\text{O} + 2.5 \text{ eV}) \]

Enthalpy of reaction per mass of reactants →

\[ u \approx 4 \text{ km/s} \]

Fuel required:

\[ \frac{|\Delta m_f|}{m_p} \approx \exp \left( \frac{\Delta v}{u} \right) \]
Fuel required: \[ \frac{|\Delta m_f|}{m_p} \approx \exp\left(\frac{\Delta v}{u}\right) \]

\( u = 4 \text{ km/s} \)

1. Trip out of solar system:
\( \Delta v_e = 12 \text{ km/s} \)

\[ \frac{|\Delta m_f|}{m_p} = e^{12/4} = 20 \]

2. Trip to Sun (0.045 AU)
\( \Delta v_e = 21 \text{ km/s} \)

\[ \frac{|\Delta m_f|}{m_p} = e^{21/4} = 190 \]
$r_p = 0.21$

**Venus**

Delta IV-Heavy: 
$\Delta v_e = 12.7 \text{ km/s}$

New Horizons

Earth

$R = 1 \text{ AU}$

Perihelion [AU]

0.21

0.046

0.723 ($R_v$)
Changing reference

Earth

Venus

R=0.72 AU

PSP

32 km/s

Venus

R=1 AU

Earth

17 km/s

PSP

0.723 \(R_v\)
$\Delta v$ for free: Venus gravity assist

R = 0.72 AU
Kepler meets Rutherford

Venusian frame

elastic collision

\[ |v_f'| = |v_i'| \]
\[ \Delta \theta = 50^\circ \]

\[ v_i' = 23 \text{ km/s} \]

\[ v_f' = 23 \text{ km/s} \]
\[ \Delta \theta = 2 \sin^{-1} \left( \frac{1}{1 + 2v_i'^2 / v_{e,p}^2} \right) \]

\[ v_i' = 23 \text{ km/s} \]

\[ \Delta \theta = 50^\circ \]

\[ \Rightarrow v_{e,p} = 28 \text{ km/s} \]

\[ \Rightarrow p = 770 \text{ km} \]

Surface of Venus @
\[ R_v = 6,000 \text{ km} \]
\[ \Delta \theta = 2 \sin^{-1} \left( \frac{1}{1 + 2 \frac{v_i'^2}{v_{e,p}^2}} \right) \]

\[ v_i' = 23 \text{ km/s} \]

\[ p > R_v = 6,000 \text{ km} \]

\[ v_{e,p} < 10.0 \text{ km/s} \]

\[ \Delta \theta < 9.7^\circ \]

\[ 50^\circ > 5 \Delta \theta \]

\[ \Rightarrow \text{6+ flybys for } r_p = 0.046 \text{ AU} \]
Why not just 6? Why the crazy spacing?
Kepler 3\textsuperscript{rd}: orbital period

\[ P \propto a^{3/2} \propto \left(2 - \frac{v^2}{v_o^2}\right)^{-3/2} \]

orbit \( b \): \( P = \frac{2}{3} P_v \)

2 orbits by Venus

3 orbits by PSP

in 450 days

return to *
outbound slingshot

\[ v_f, v_i, v_v, v_r, v_{\phi}, \Delta v \]
in-in or out-out rendezvous:
\[ m P_v - \delta_v = n P_{sc} - \delta_{sc} \]

out-in rendezvous:
\[ m P_v + \delta_v = n P_{sc} + \delta_{sc} \]

resonance:
\[ m P_v = n P_{sc} \]
orbits allowing repeated flybys after $\Delta t \leq 3 \, P_v$
legal moves: \[
\begin{align*}
\text{in-out} & \rightarrow \text{out-in} \quad [\text{via out-out (res)}] \\
\text{out-in} & \rightarrow \text{in-out} \quad [\text{via in-in (res)}]
\end{align*}
\]
239 days

2226 days = 6 Earth years
day 0.26
κ = 0.65
ι = 0.60 \, ι_a
orbit 1
2018-08-12
destination: orbit b

\( r_a = 0.94 \text{ AU} \)

\( r_p = 0.16 \text{ AU} \)

1st orbit: a

\( P = \frac{2}{3} P_v \)

\( \Delta \theta = 9.7^\circ \)

\( v'_i = 23 \text{ km/s} \)
$0.23 \text{ AU}$

$0.19 \text{ AU}$

$1.08 \text{ AU}$

$\Delta \theta = 9.7^\circ$

$1^{\text{st orbit}} : a$

launch
Launch criteria:

- \( \Delta v_e < 12.8 \text{ km/s} \)
- \( v'_i = 23 \text{ km/s} \)
- \( \Delta \theta < 9.7^\circ \)
Venus lags Earth by 46° on:

- 2015-06-05
- 2017-01-09
- 2018-08-12
- 2020-03-20

19 months apart

Center of the Launch window:

35 km/s

51 days = 81°
Which Planets are Good?

\[ \Delta \theta = 2 \sin^{-1}\left( \frac{1}{1 + 2 \frac{v_i'}{v_{e,p}}^2} \right) \]

\[ v_i' = \Delta v \sim v_o \]

characteristic scattering angle

\[ \Delta \theta_{ch} = 2 \sin^{-1}\left( \frac{1}{1 + 2 \left( \frac{v_o}{v_{e,s}} \right)^2} \right) \]

surface escape speed

planetary orbital speed

\[ v_o = \sqrt{\frac{GM}{R}} \]
Which Planets are Good?

characteristic scattering angle

\[ \Delta \theta_{ch} = 2 \sin^{-1} \left[ \frac{1}{1 + 2 \left( \frac{v_{esc,s}}{v_o} \right)^2} \right] \]

<table>
<thead>
<tr>
<th></th>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
<th>Jupiter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_o ) [km/s]</td>
<td>48</td>
<td>35</td>
<td>30</td>
<td>24</td>
<td>13</td>
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<tr>
<td>( V_{esc,s} ) [km/s]</td>
<td>4.2</td>
<td>10.4</td>
<td>11.2</td>
<td>5.0</td>
<td>60</td>
</tr>
<tr>
<td>( V_{esc,s} / V_o )</td>
<td>0.087</td>
<td>0.28</td>
<td>0.37</td>
<td>0.21</td>
<td>4.6</td>
</tr>
<tr>
<td>( \Delta \theta_{ch} )</td>
<td>0.4°</td>
<td>4.5°</td>
<td>7.2°</td>
<td>2.4°</td>
<td>132°</td>
</tr>
</tbody>
</table>
Summary

• Launch speed (rel. to Earth) $\Delta v_e$ limited by rocket
• Can get extra $\Delta v$ from gravity assist: scattering off moving scatterer (planet)
• Elastic scattering $\rightarrow |\Delta v_p|$ conserved
• Scattering angle = Rutherford scattering angle
• Limited by surface escape speed
• May need multiple scatterings (grav. assists) to achieve desired orbit
• Multiple assists require resonant orbits, in-out or out-in rendezvous orbits