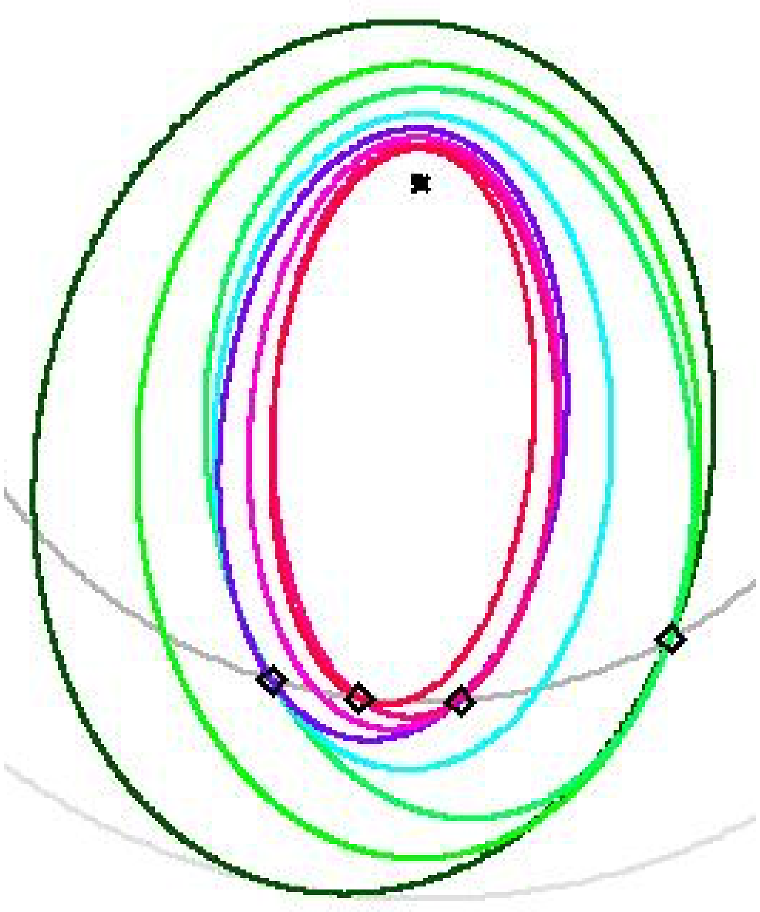
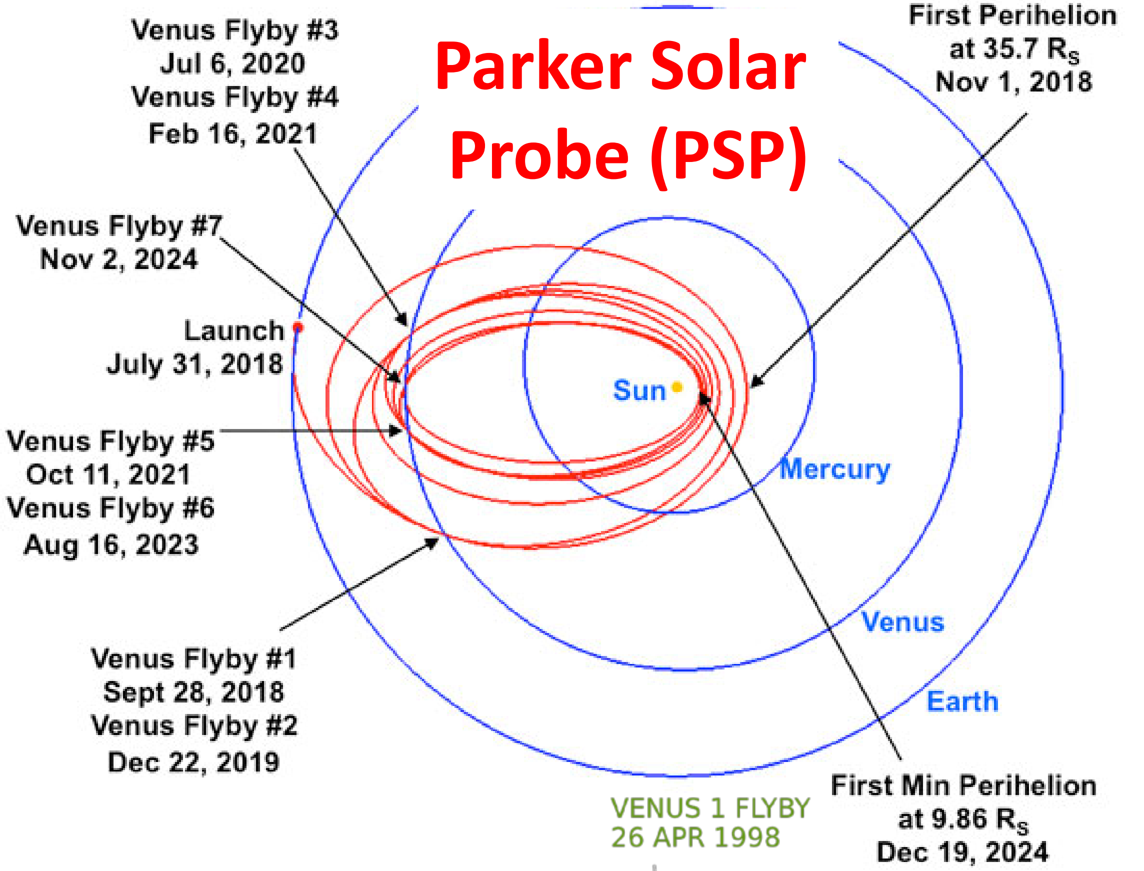
How to get anywhere in the Solar System on a single tank of fuel



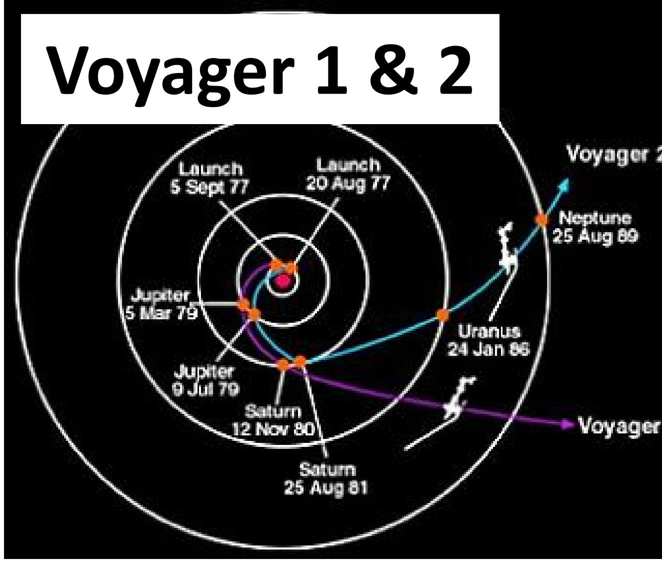
A dilettante dabbles in orbital mechanics

Dana Longcope

Montana State University



Parker Solar Probe (PSP)



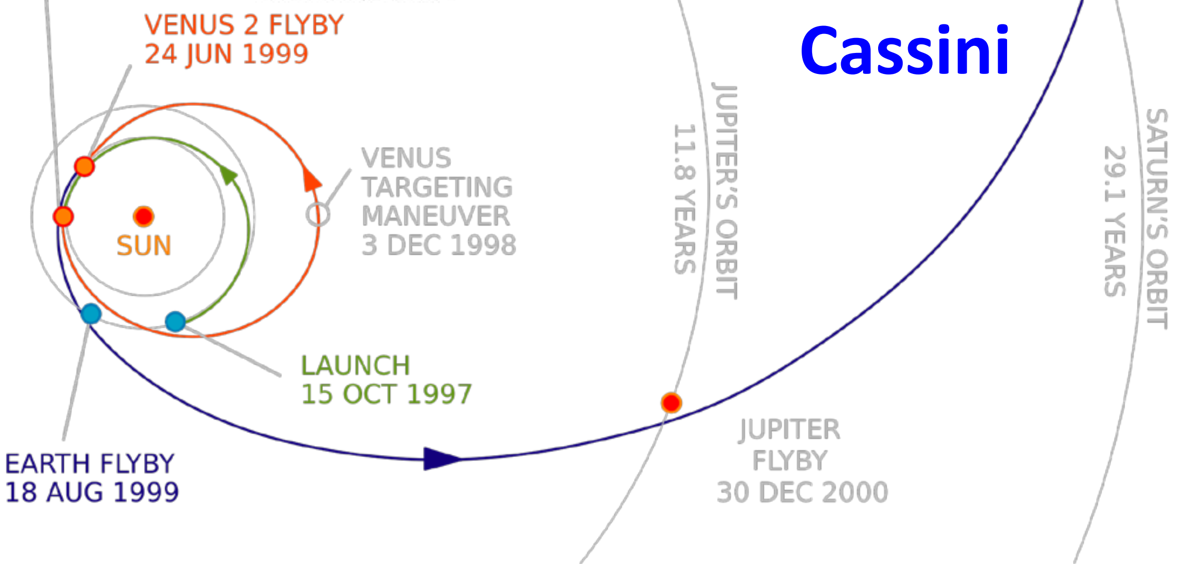
Voyager 1 & 2

Gravity Assists:

Why?

How?

Why so many?



Cassini

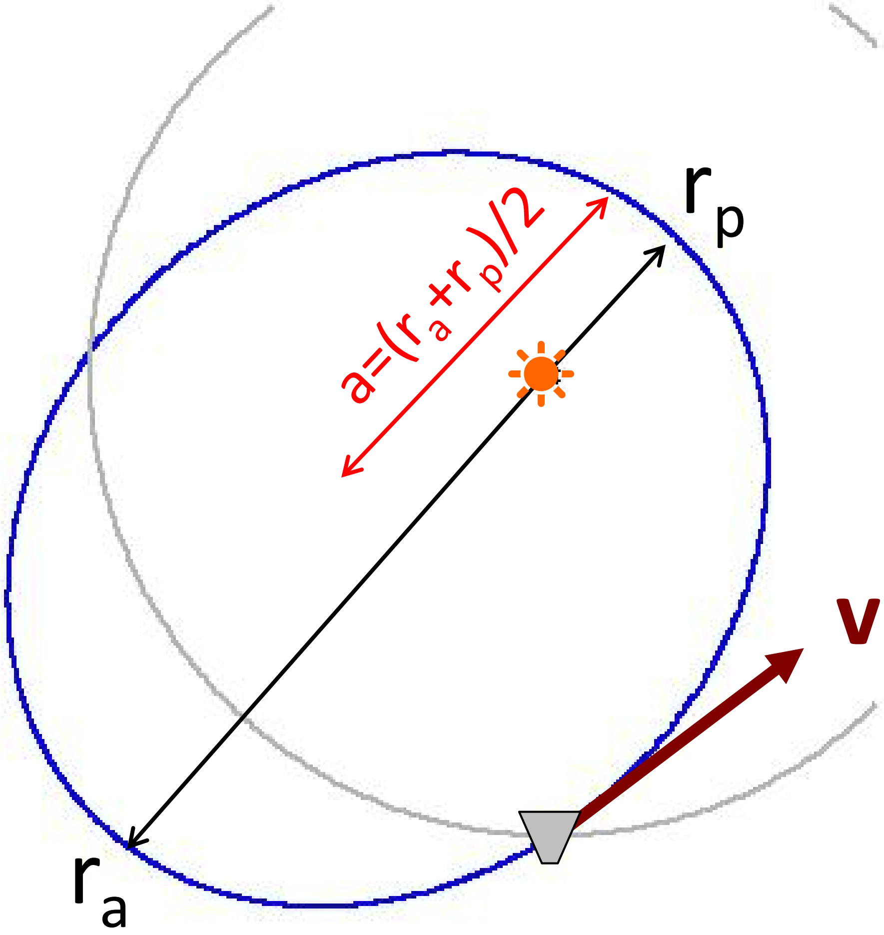
From Fran Bagenal’s lecture yesterday:

Sample Trajectory to Uranus:
Earth-Venus-Earth-Earth-Jupiter
Gravity Assist

… say what!?

eccentricity: $$e=\frac{r_a-r_p}{r_a+r_p}$$

$$r_a=(1+e)a \qquad r_p=(1-e)a$$



S/C orbits Sun
ballistically:

ellipse w/
Sun @ focus

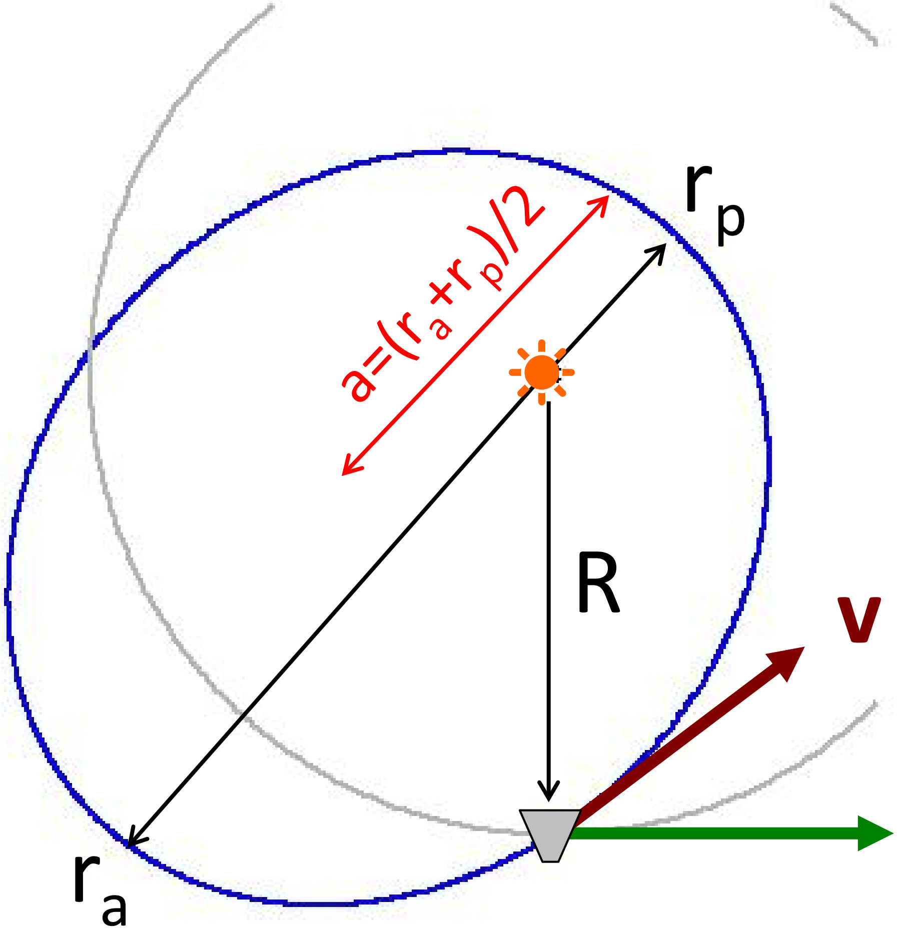
(Kepler's 1st)

eccentricity: $e = \dfrac{r_a - r_p}{r_a + r_p}$

$$E = \tfrac{1}{2}\mathrm{v}^2 - \frac{GM}{r} = -\frac{GM}{2a}$$

$r_a = (1+e)a \qquad r_p = (1-e)a$

$$\boxed{\frac{a}{R} = \frac{1}{2 - \mathrm{v}^2/\mathrm{v}_o^2}}$$



$$p_\varphi^2 = r^2\mathrm{v}_\varphi^2 = GMa(1-e^2)$$

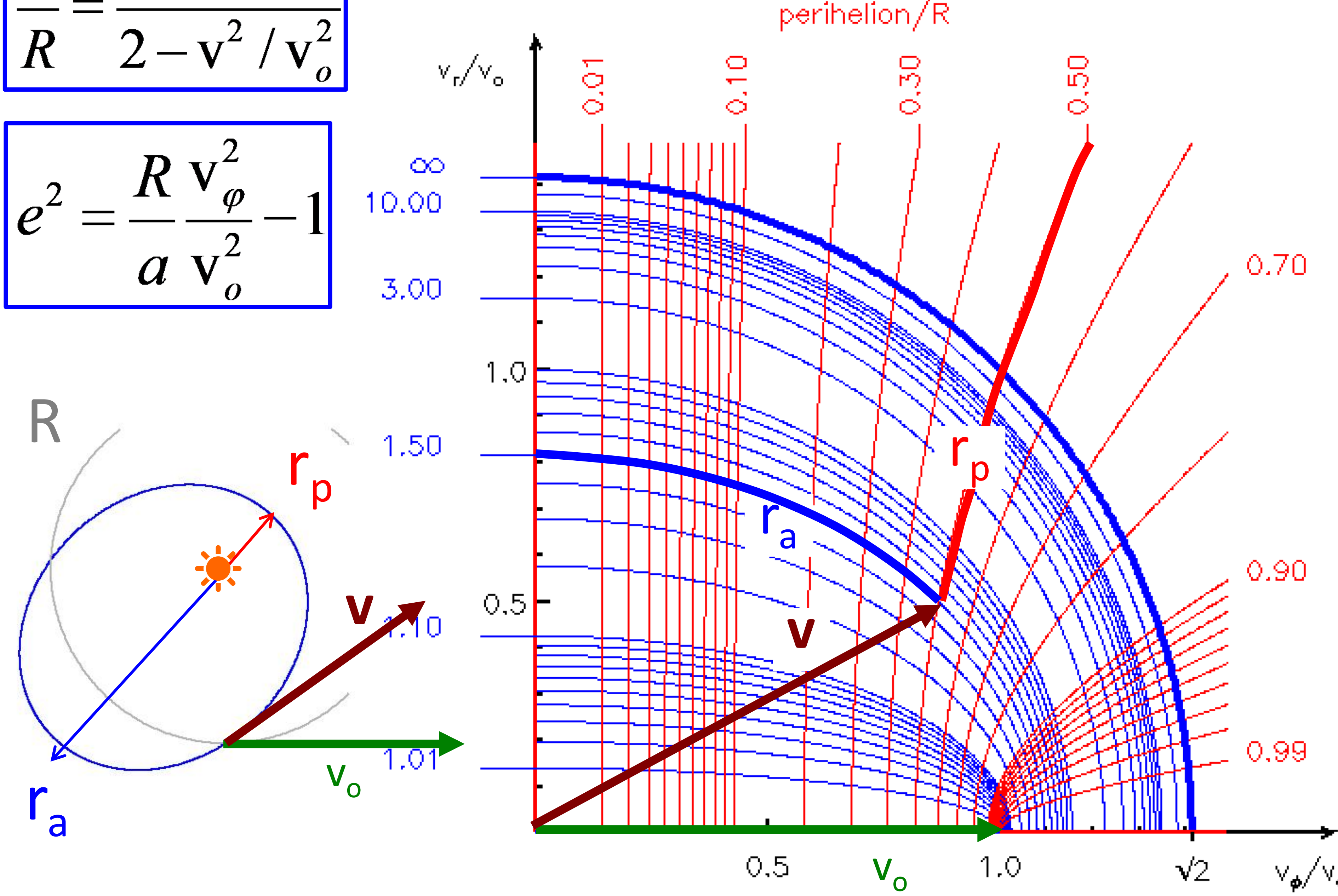
$$\boxed{e^2 = \frac{R}{a}\frac{\mathrm{v}_\varphi^2}{\mathrm{v}_o^2} - 1}$$

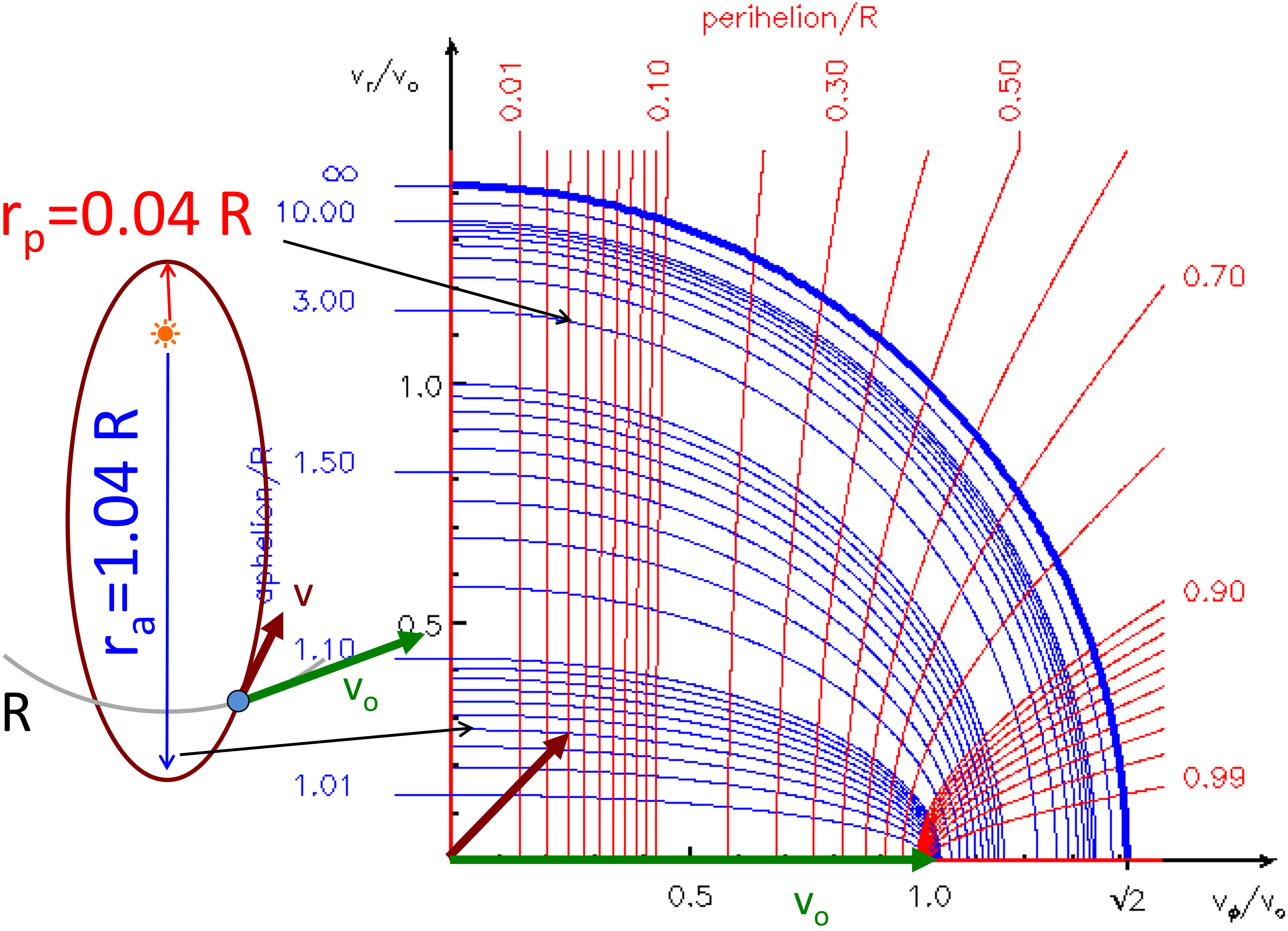
$$\mathrm{v}_o = \sqrt{\frac{GM}{R}}$$

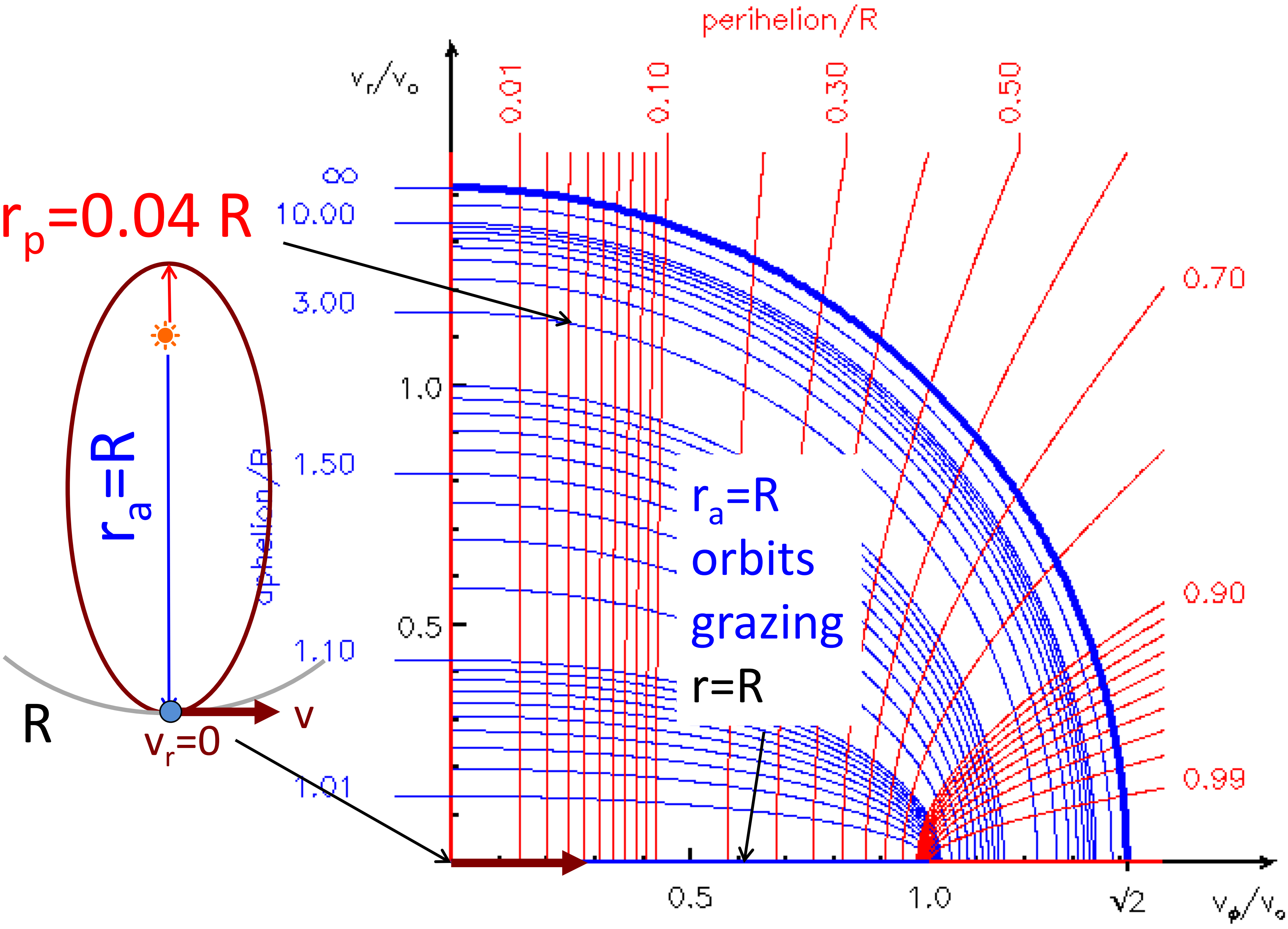
$$\frac{a}{R} = \frac{1}{2 - \mathrm{v}^2 / \mathrm{v}_o^2}$$

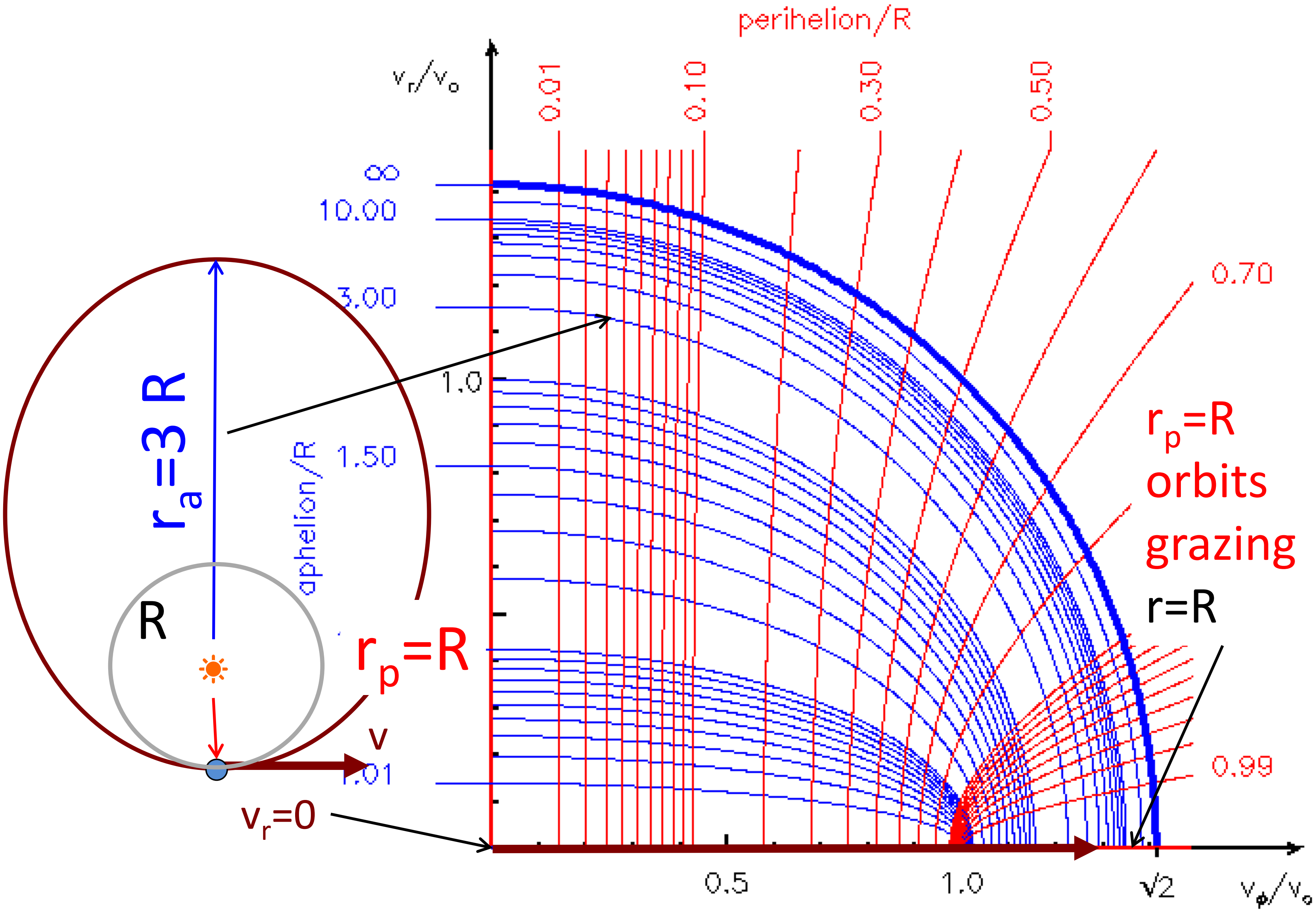
$$e^2 = \frac{R}{a}\frac{\mathrm{v}_\varphi^2}{\mathrm{v}_o^2} - 1$$

$$r_a = (1+e)a \qquad r_p = (1-e)a$$

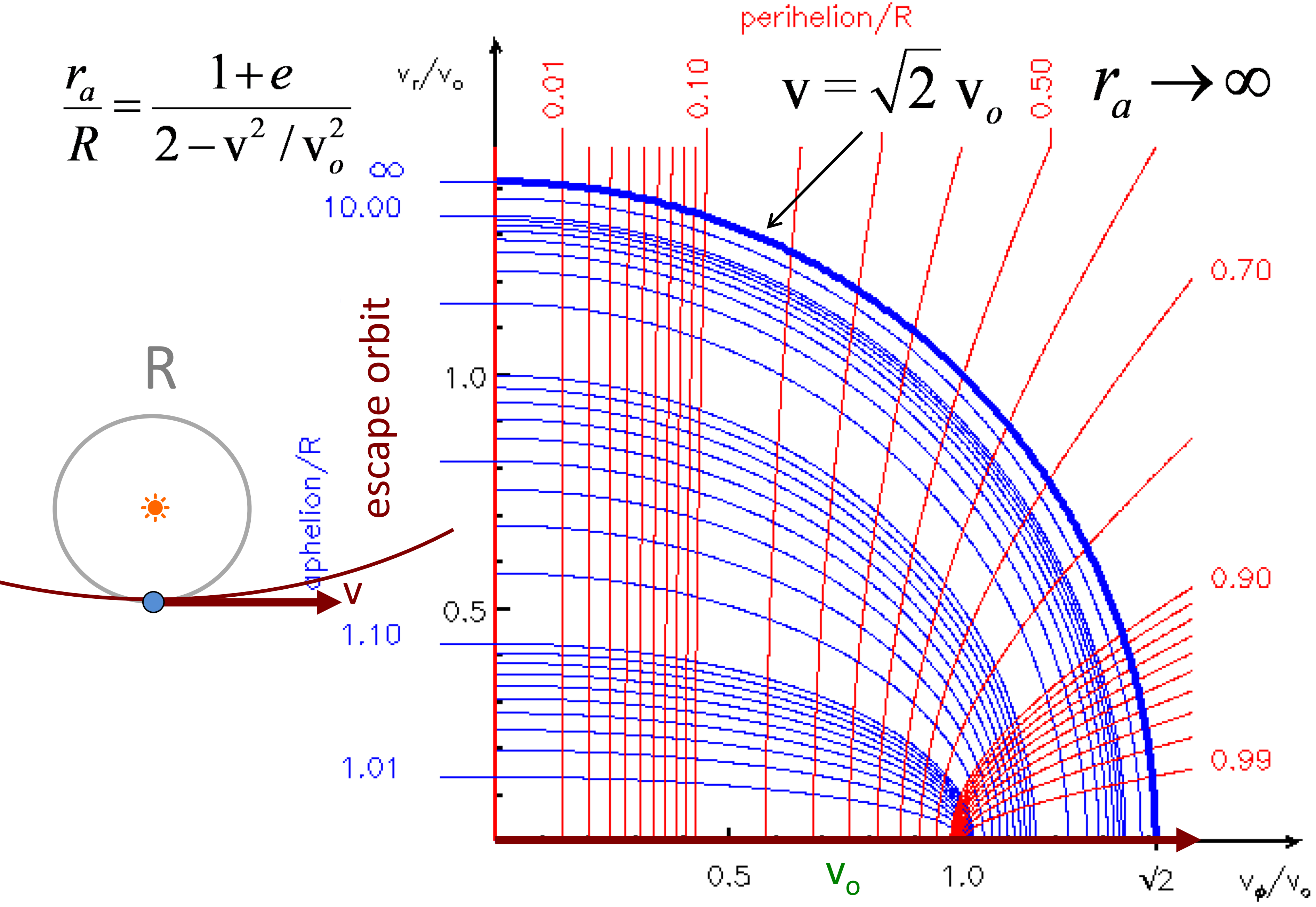



perihelion/R
v_r/v_o
0.01
0.10
0.30
0.50
0.70
0.90
0.99
∞
10.00
3.00
1.50
1.10
1.01
aphelion/R
1.0
0.5
0.5
1.0
√2
v_ϕ/v_o
r_p=0.04 R
r_a=1.04 R
v
v_o
v_o
R




perihelion/R
v_r/v_o
0.01
0.10
0.30
0.50
0.70
0.99
∞
10.00
3.00
1.0
1.50
1.01
aphelion/R
0.5
1.0
√2
v_φ/v_o
r_a=3 R
R
r_p=R
v
v_r=0
r_p=R
orbits
grazing
r=R

$$\frac{r_a}{R} = \frac{1+e}{2 - v^2 / v_o^2}$$



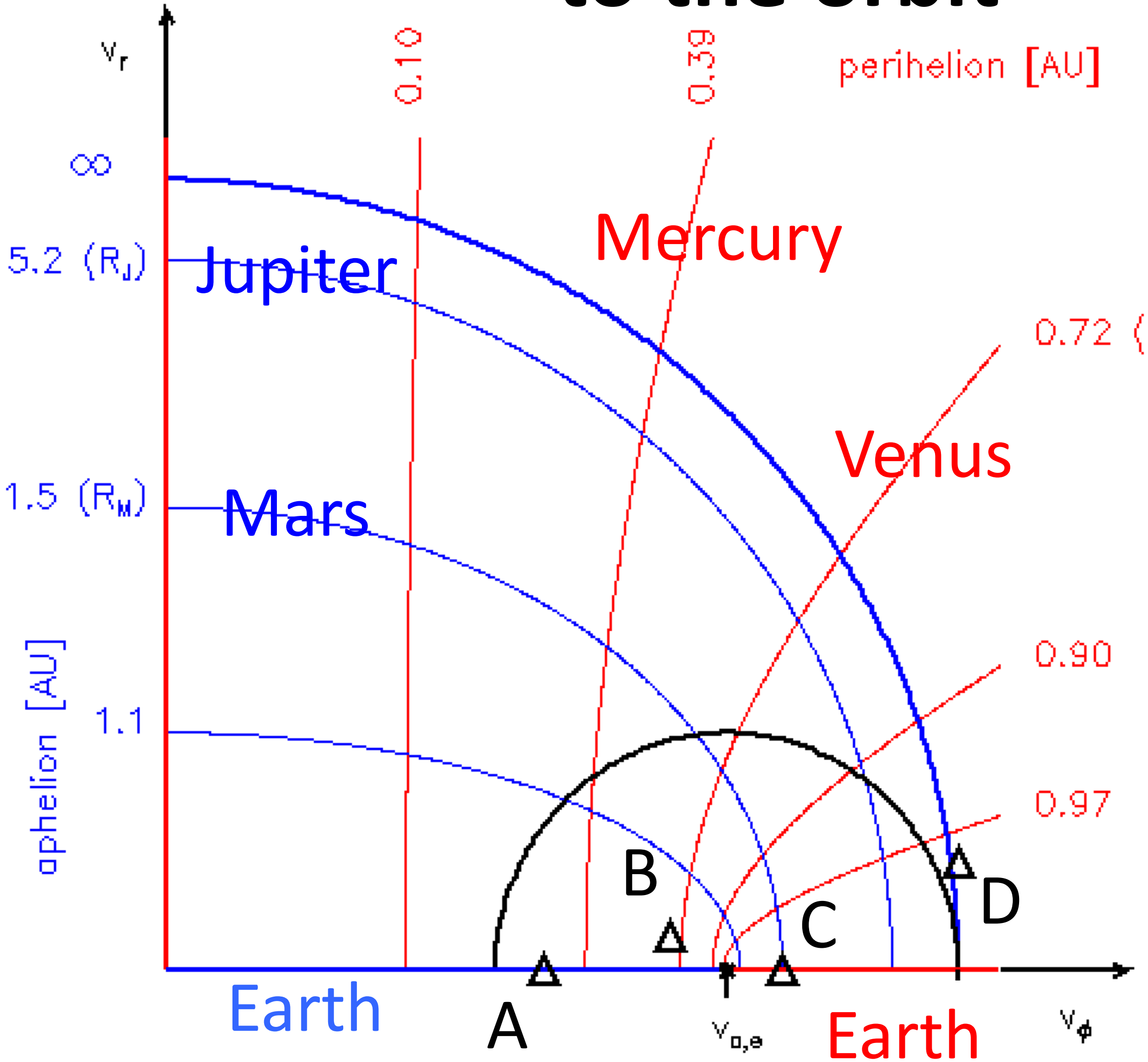
$$v = \sqrt{2}\, v_o$$

$$r_a \rightarrow \infty$$

escape orbit

R=1 AU

Match the mission to the orbit



☐ MAVEN:

Earth → Mars direct

☐ Helios 2:

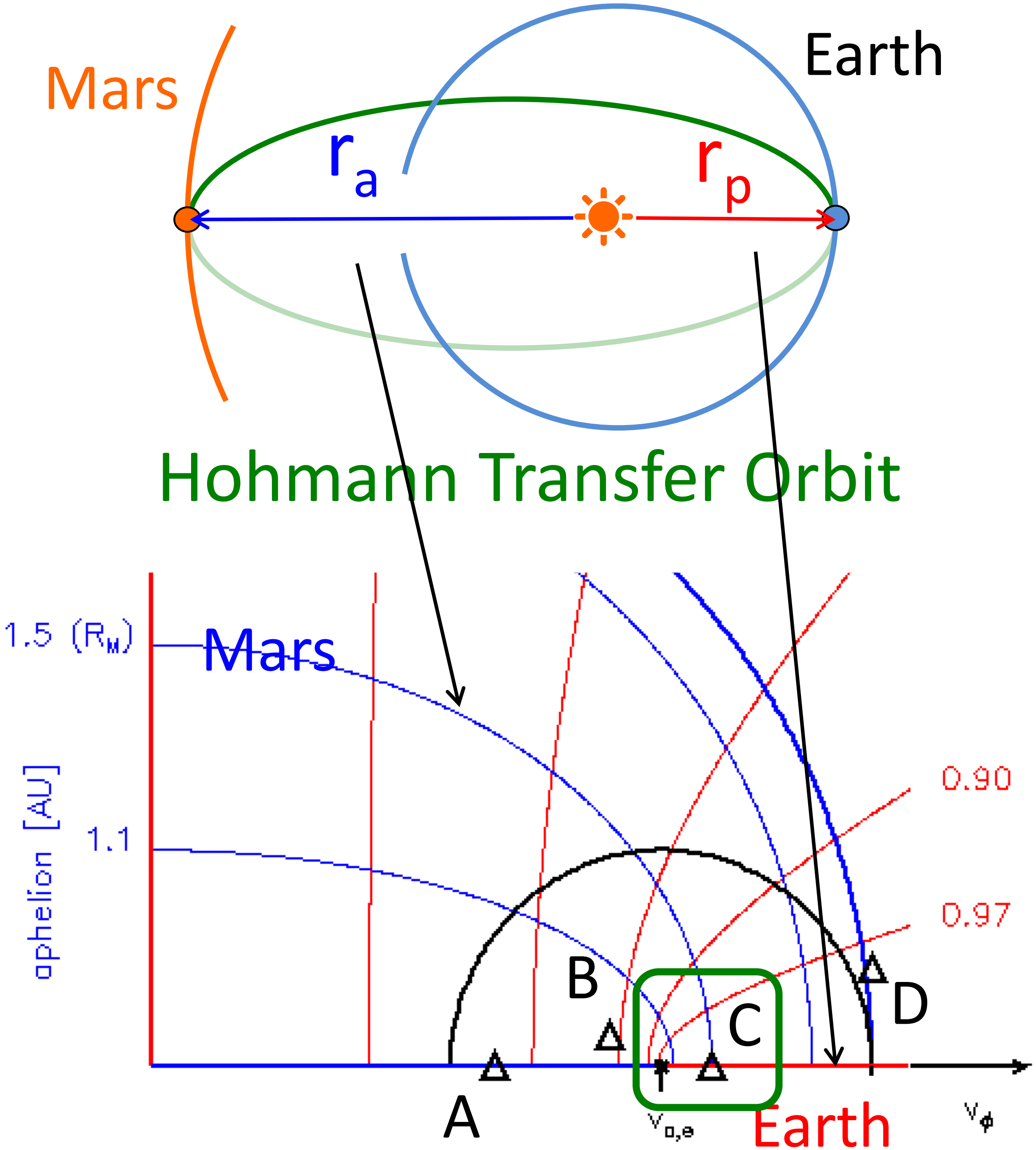
1976: sampled solar wind 0.29 < r < 1 AU

☐ New Horizons:

Earth → Jupiter → out passing Pluto & Ultima Thule

☐ Cassini:

Earth → Venus → Earth → Venus → Saturn

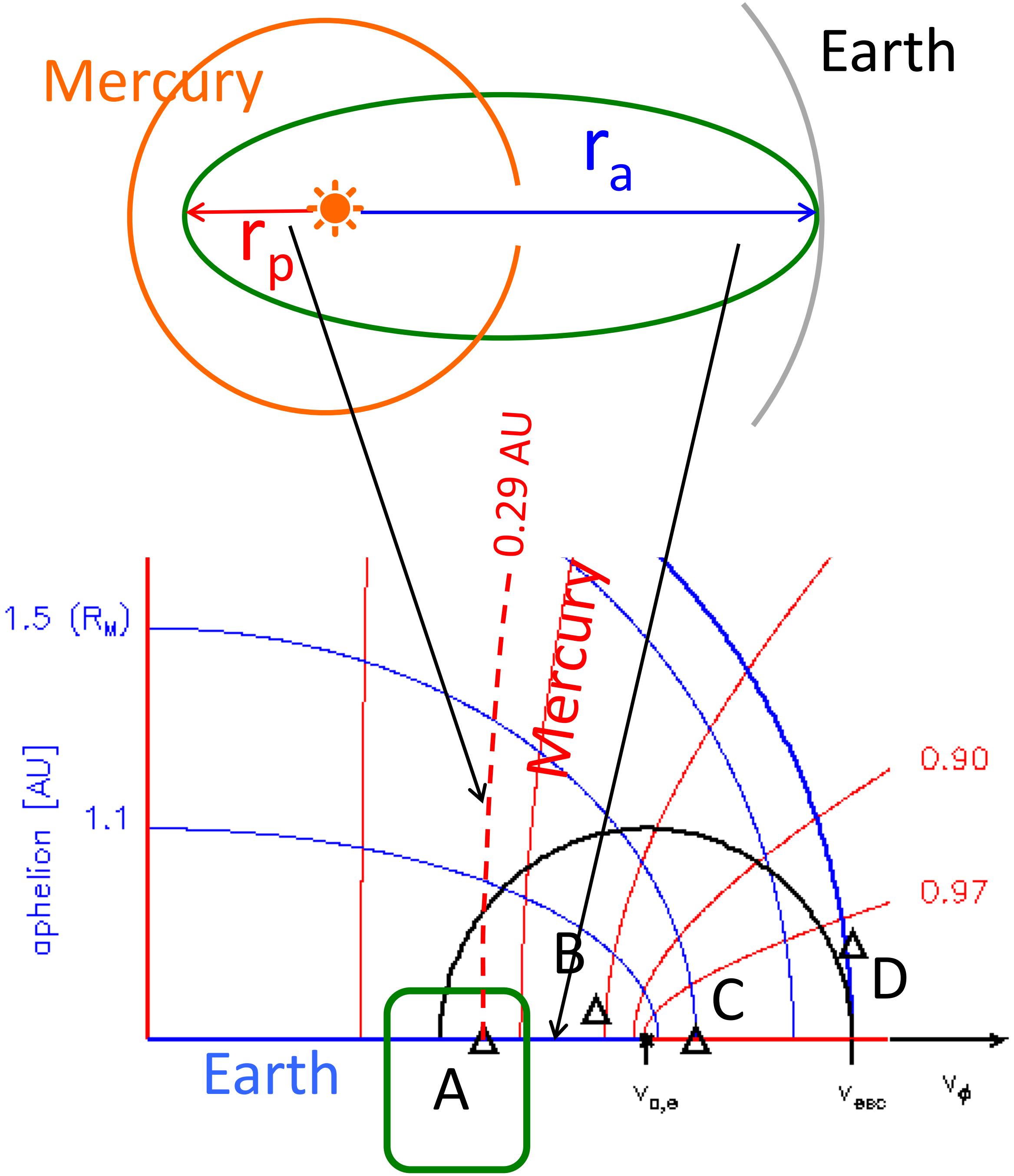


C MAVEN:
Earth → Mars direct

☐ Helios 2:
1976: sampled solar wind 0.29 < r < 1 AU

☐ New Horizons:
Earth → Jupiter → out passing Pluto & Ultima Thule

☐ Cassini:
Earth → Venus → Earth → Venus → Saturn



C MAVEN:

Earth → Mars direct

A Helios 2:

1976: sampled solar wind 0.29 < r < 1 AU

☐ New Horizons:

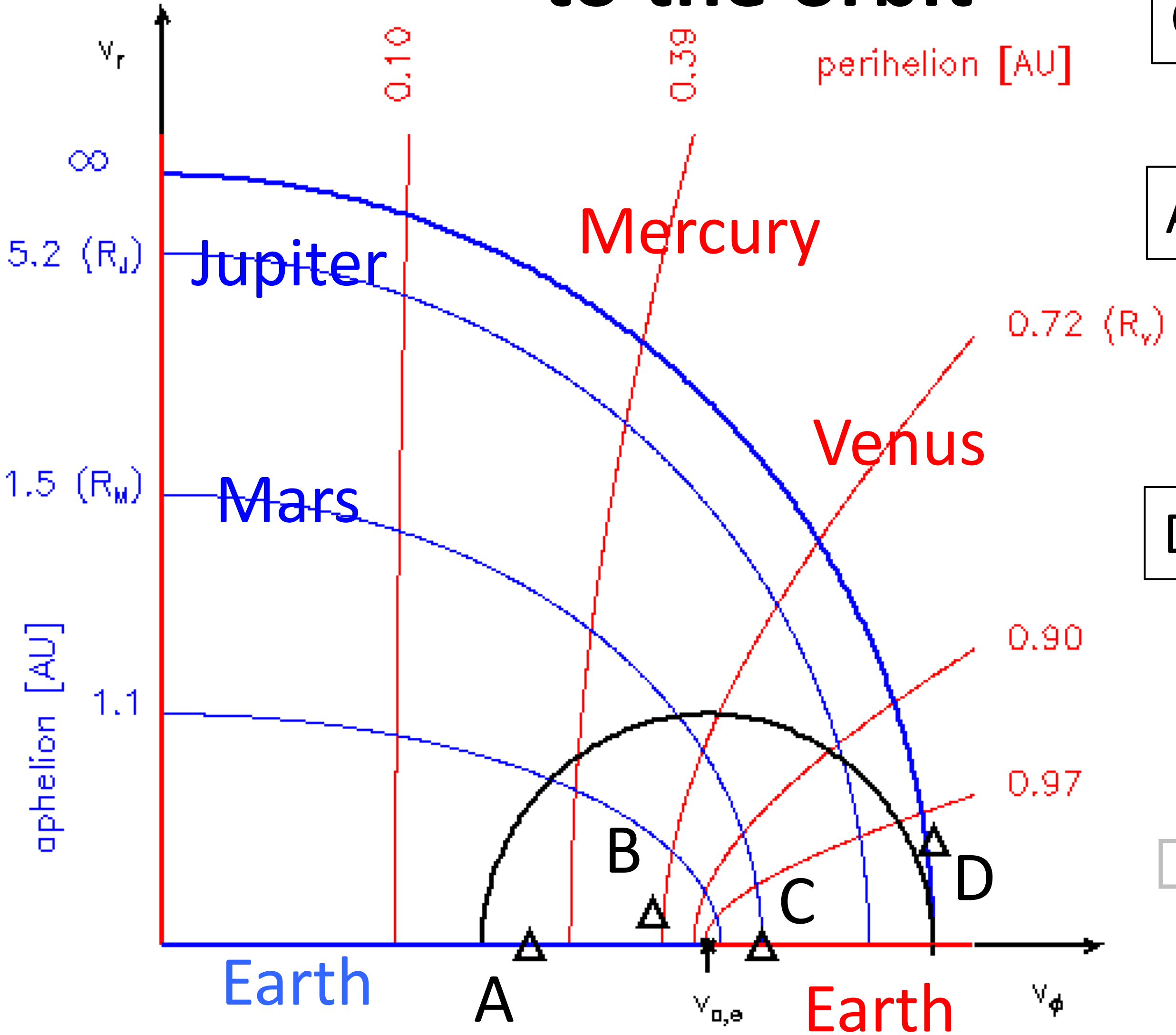
Earth → Jupiter → out passing Pluto & Ultima Thule

☐ Cassini:

Earth → Venus → Earth → Venus → Saturn

R=1 AU

Match the mission to the orbit



C **MAVEN:**

Earth → Mars direct

A **Helios 2:**

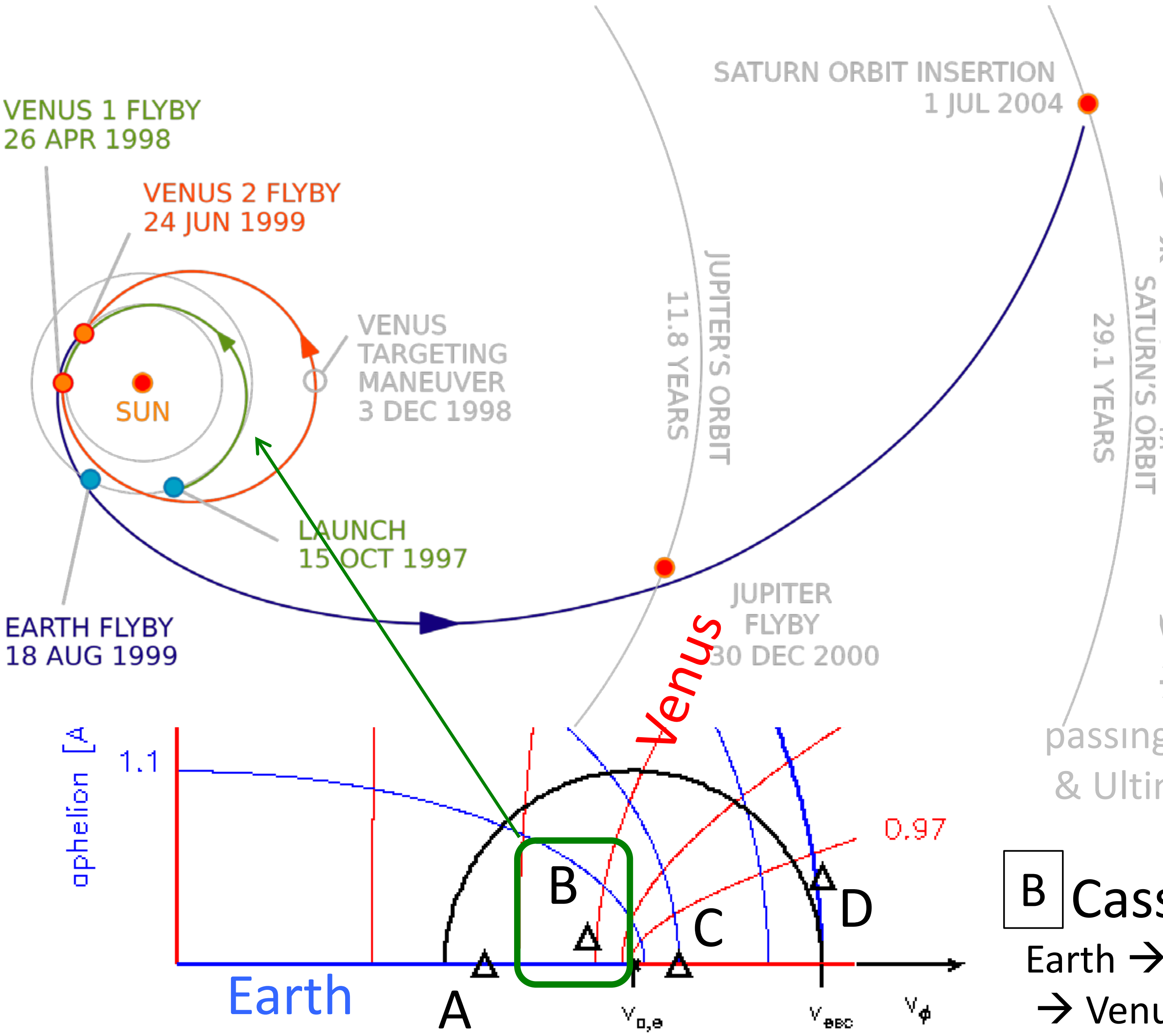
1976: sampled solar wind 0.29 < r < 1 AU

D **New Horizons:**

Earth → Jupiter → out passing Pluto & Ultima Thule

☐ **Cassini:**

Earth → Venus → Earth → Venus → Saturn



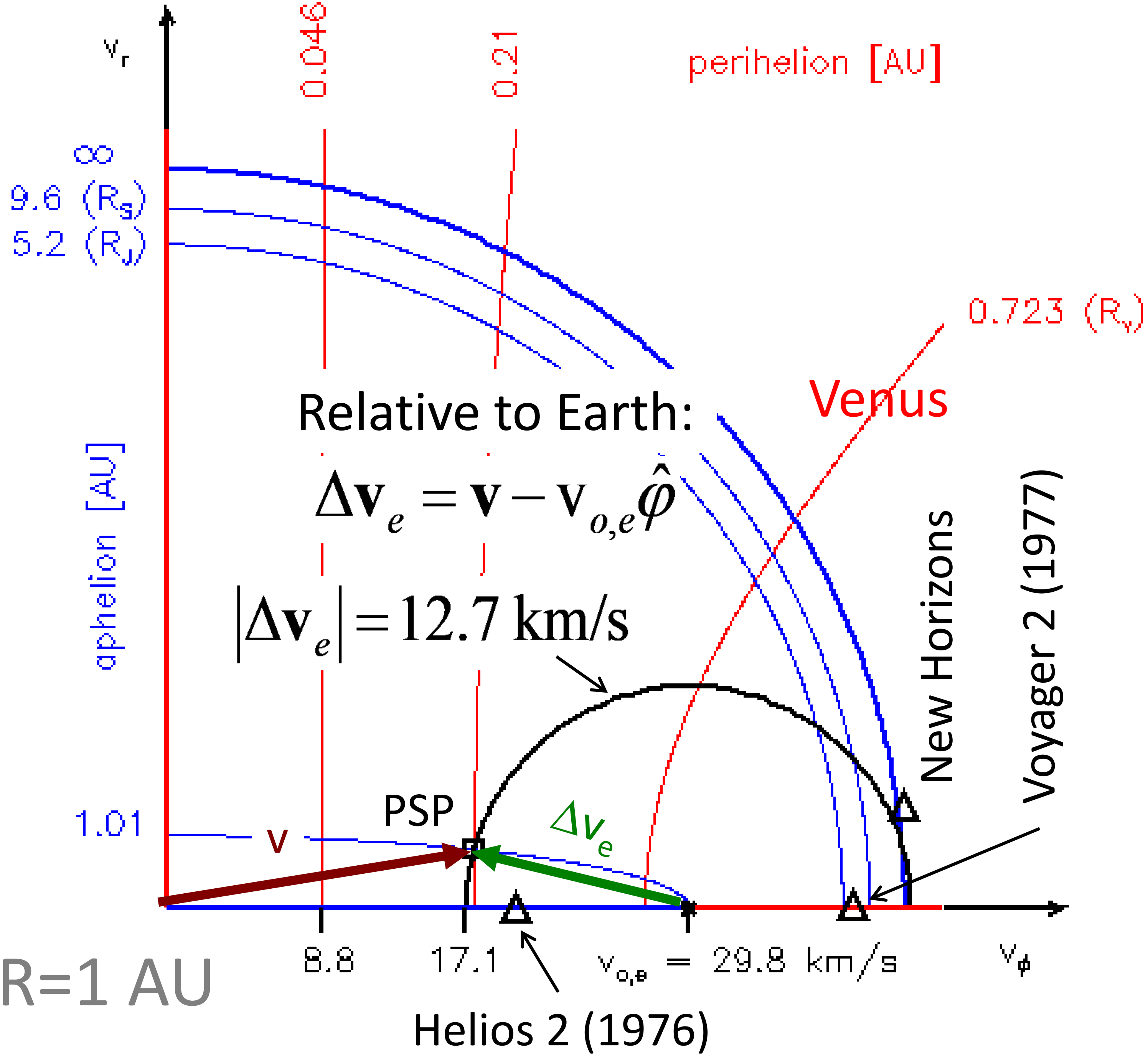
VEN:
→ Mars direct

os 2:
sampled solar
0.29 < r < 1 AU

v Horizons:
→ Jupiter → out
passing Pluto
& Ultima Thule

B Cassini:

Earth → Venus → Earth
→ Venus → Saturn



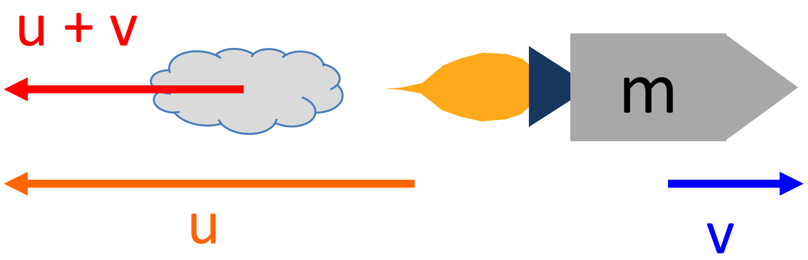
Relative to Earth:

$$\Delta\mathbf{v}_e = \mathbf{v} - \mathbf{v}_{o,e}\hat{\varphi}$$

$$|\Delta\mathbf{v}_e| = 12.7\text{ km/s}$$

R=1 AU

$$m\frac{d\mathrm{v}}{dt} = F = |\dot{m}|u$$

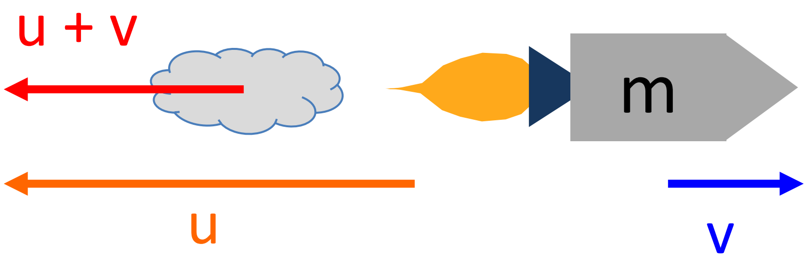


$$\boxed{\Delta\mathrm{v} = u\ln\left(\frac{|\Delta m_f| + m_p}{m_p}\right) \approx u\ln\left(\frac{|\Delta m_f|}{m_p}\right)}$$

Impulse: $\int F\,dt = \int u\,\dot{m}\,dt = u\,|\Delta m|$

Specific impulse: $u = \dfrac{\int F\,dt}{|\Delta m|}$ [m/s] momentum/mass

$$m\frac{d\mathrm{v}}{dt} = F = |\dot{m}|u$$



$$\boxed{\Delta \mathrm{v} \approx u \ln\left(\frac{|\Delta m_f|}{m_p}\right)}$$

depends on:

1. mass ratio fuel : payload
2. specific impulse u

Specific impulse depends on fuel:

$2H_2 + O_2 \rightarrow 2 \times (H_2O + 2.5\ \mathrm{eV})$

Enthalpy of reaction per mass of reactants ➔

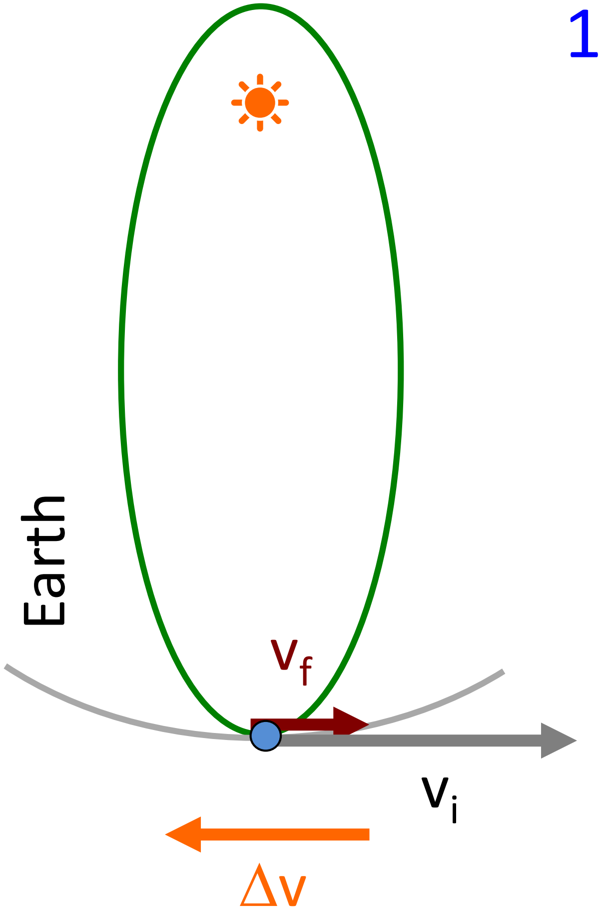
$u \simeq 4$ km/s

Fuel required: $$\frac{|\Delta m_f|}{m_p} \approx \exp\left(\frac{\Delta \mathrm{v}}{u}\right)$$

Fuel required:

$$\frac{|\Delta m_f|}{m_p} \approx \exp\left(\frac{\Delta \mathrm{v}}{u}\right)$$

u = 4 km/s



1. Trip out of solar system:

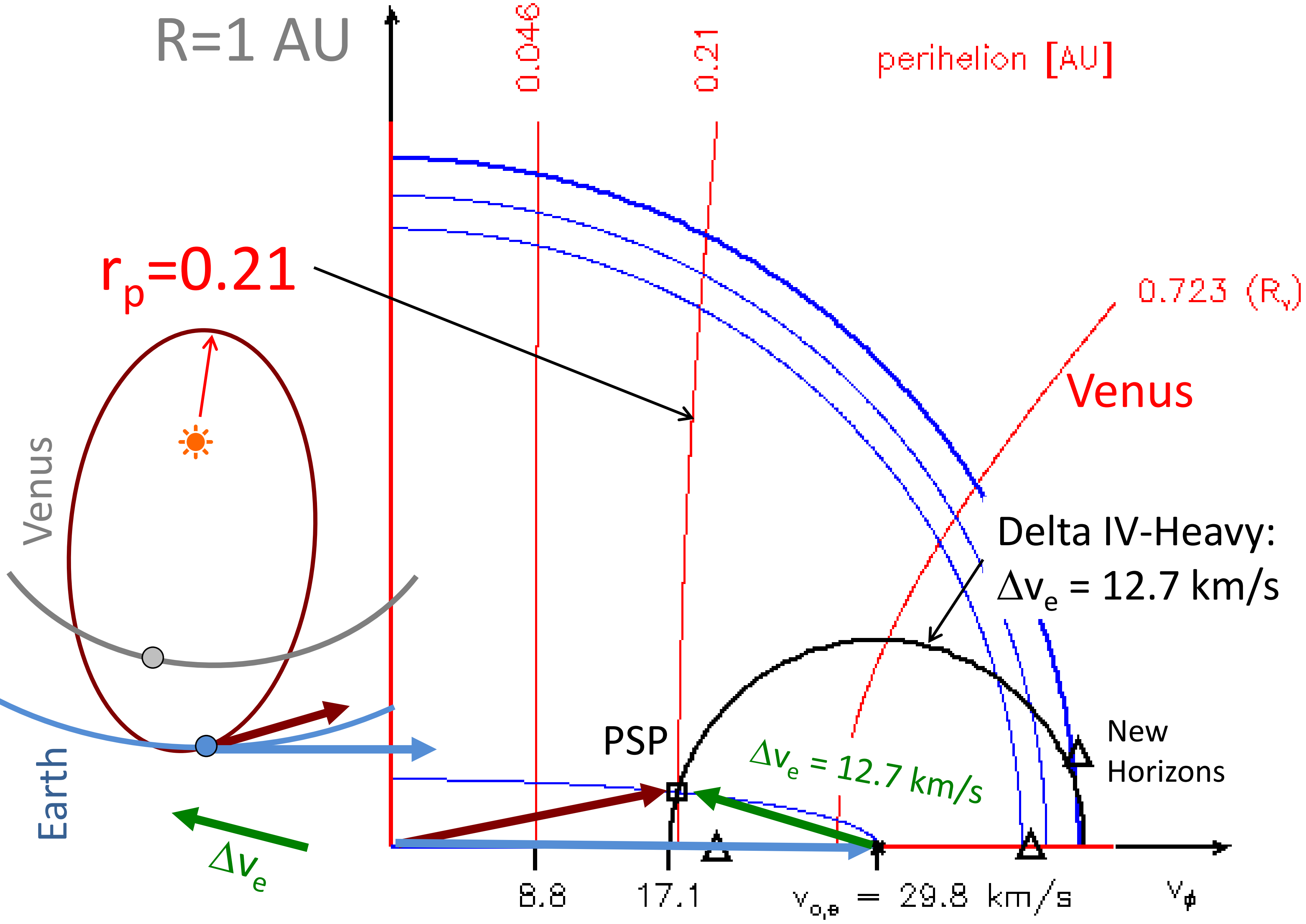
Δv_e = 12 km/s

$$\frac{|\Delta m_f|}{m_p} = e^{12/4} = \quad 20$$

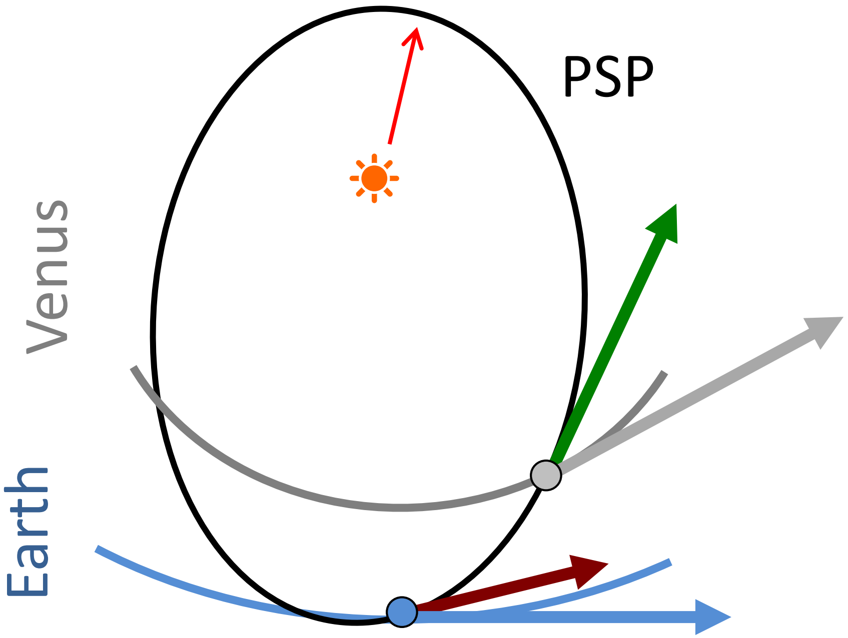
2. Trip to Sun (0.045 AU)

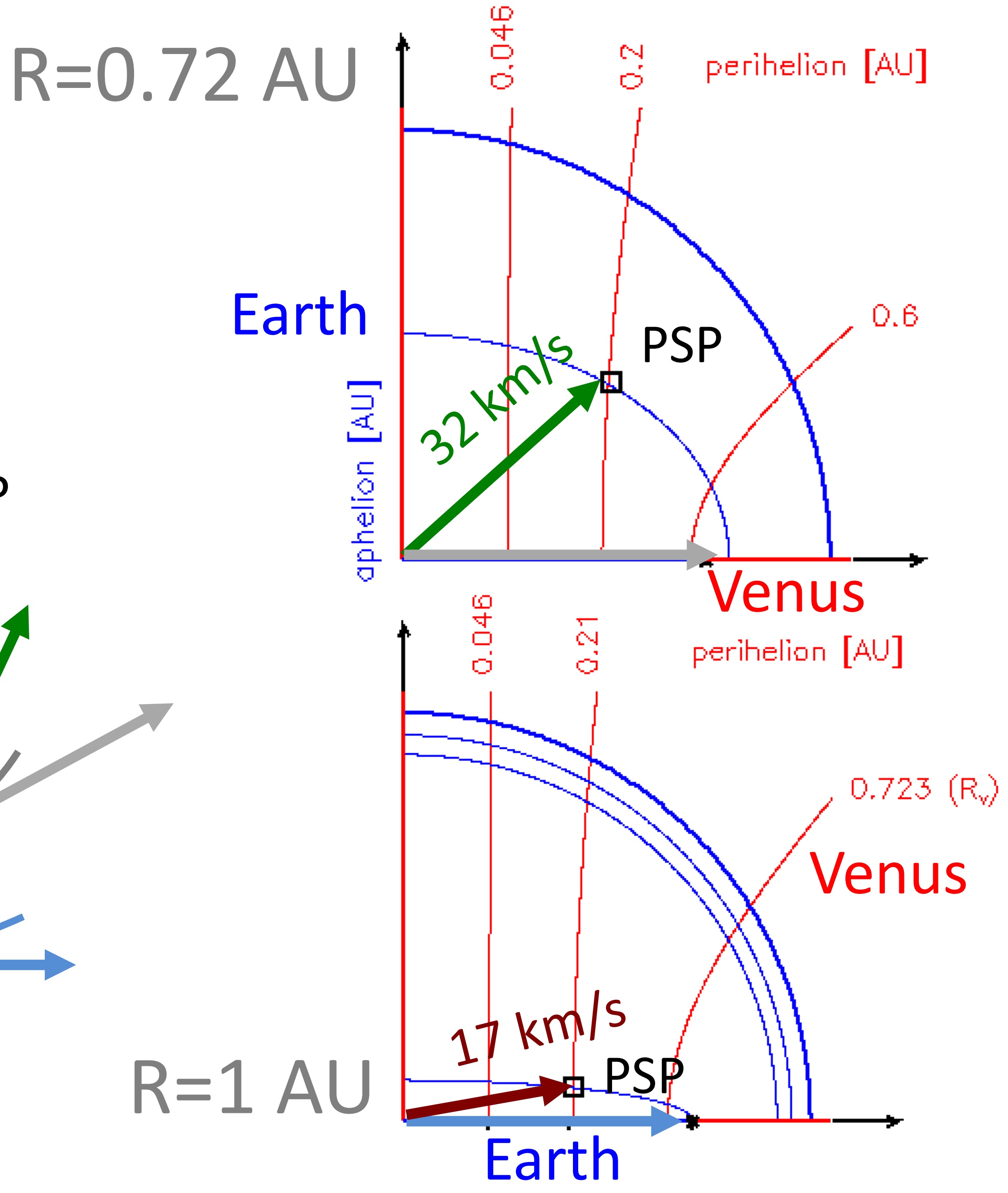
Δv_e = 21 km/s

$$\frac{|\Delta m_f|}{m_p} = e^{21/4} = \quad 190$$


R=1 AU
0.046
0.21
perihelion [AU]
r_p=0.21
0.723 (R_v)
Venus
Venus
Delta IV-Heavy:
Δv_e = 12.7 km/s
PSP
Δv_e = 12.7 km/s
New Horizons
Earth
Δv_e
8.8
17.1
$v_{o,e}$ = 29.8 km/s
v_ϕ

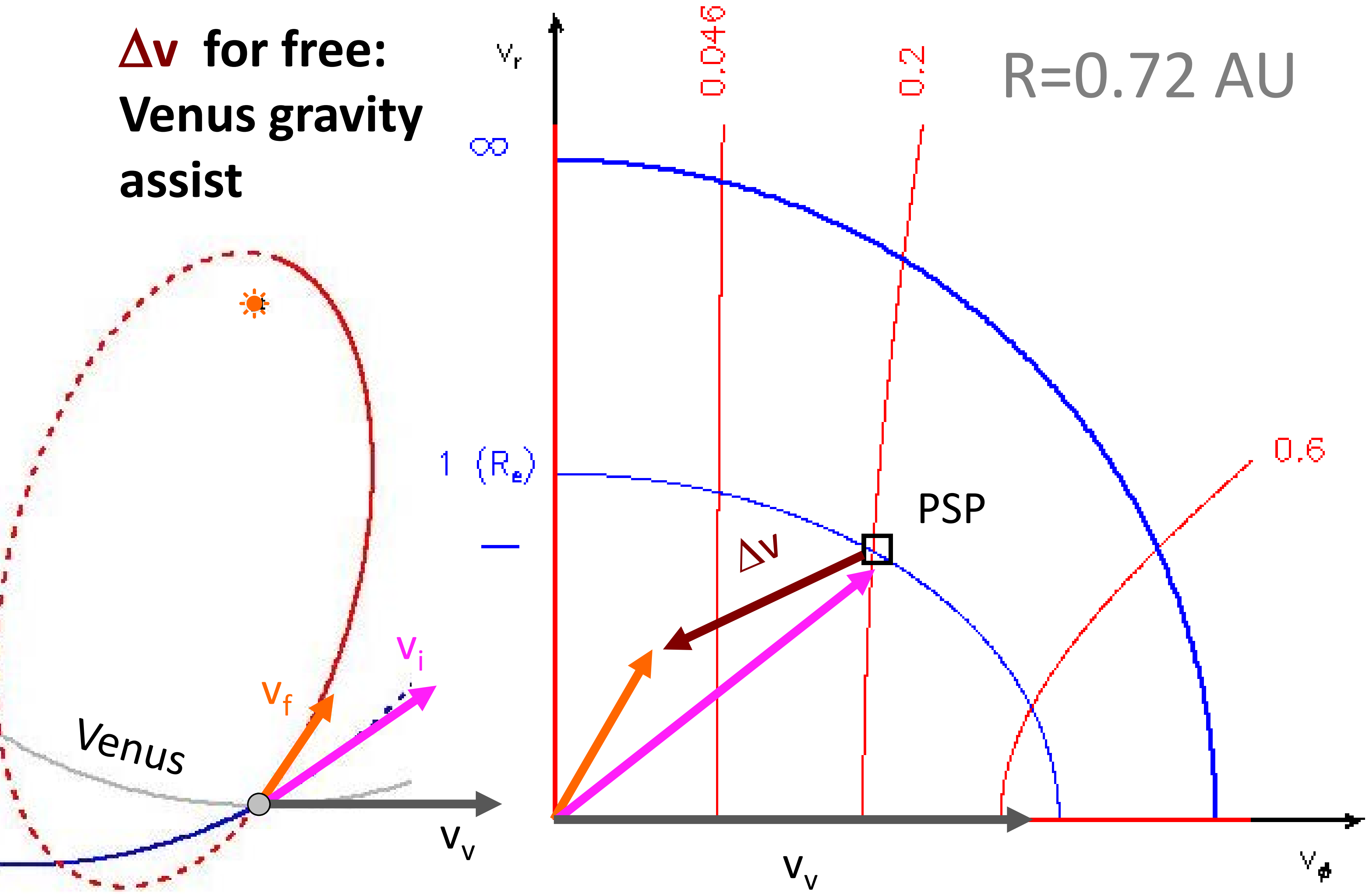
Changing reference



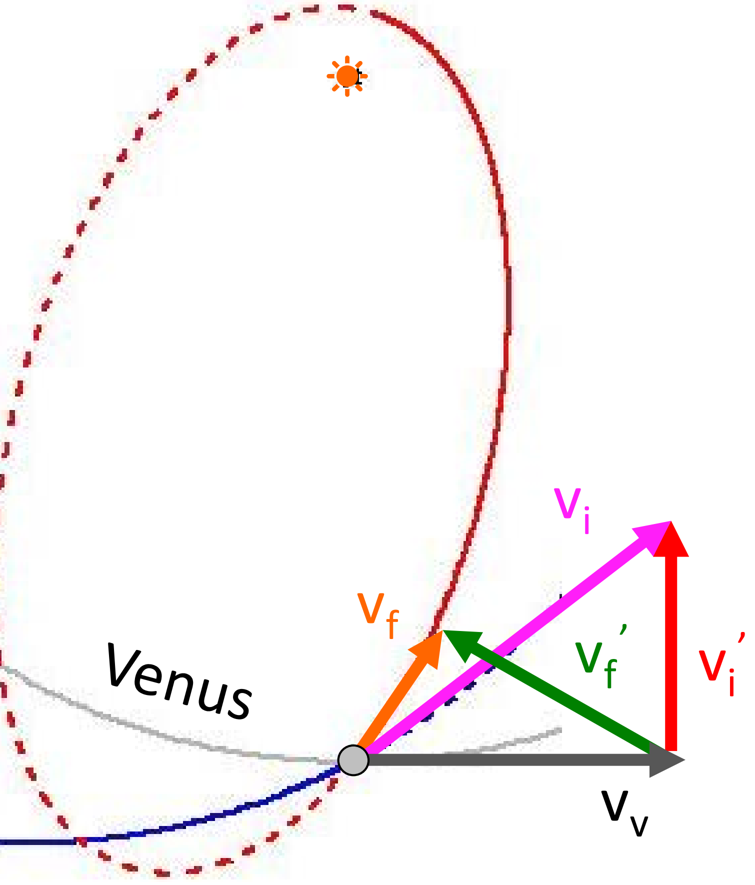


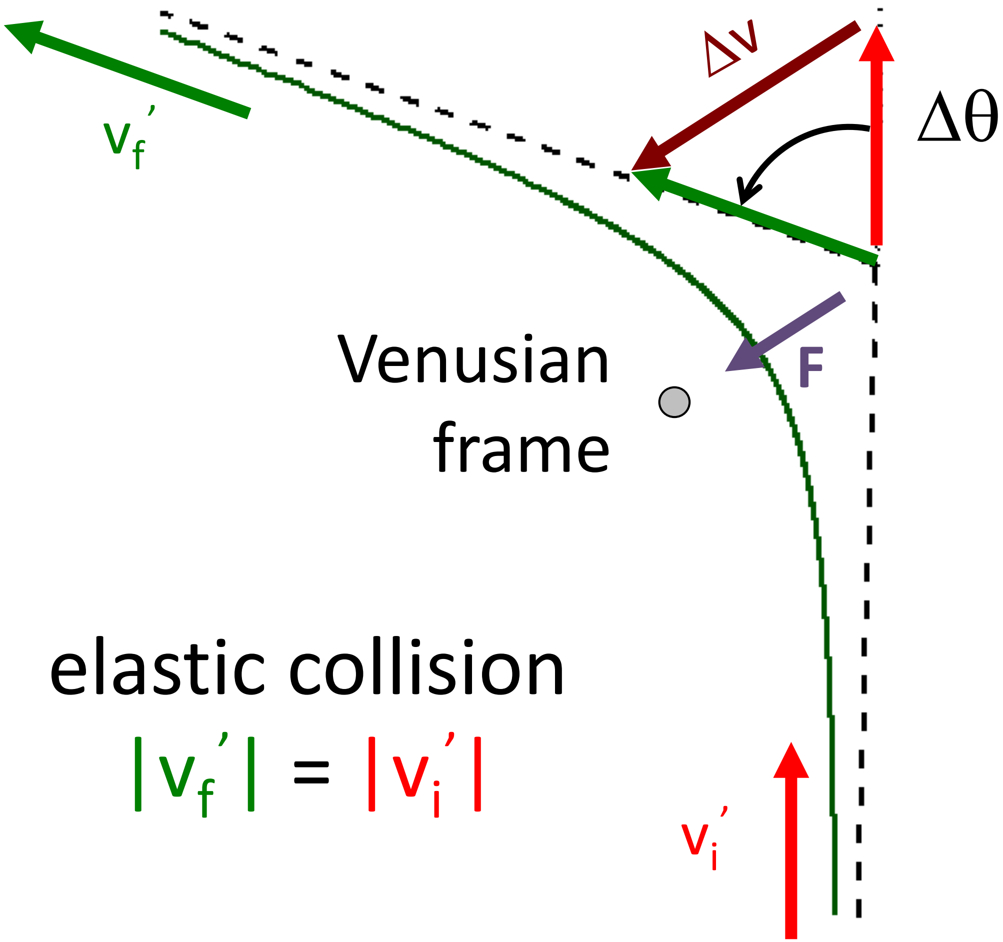
Δv for free: Venus gravity assist

R=0.72 AU



Kepler meets Rutherford



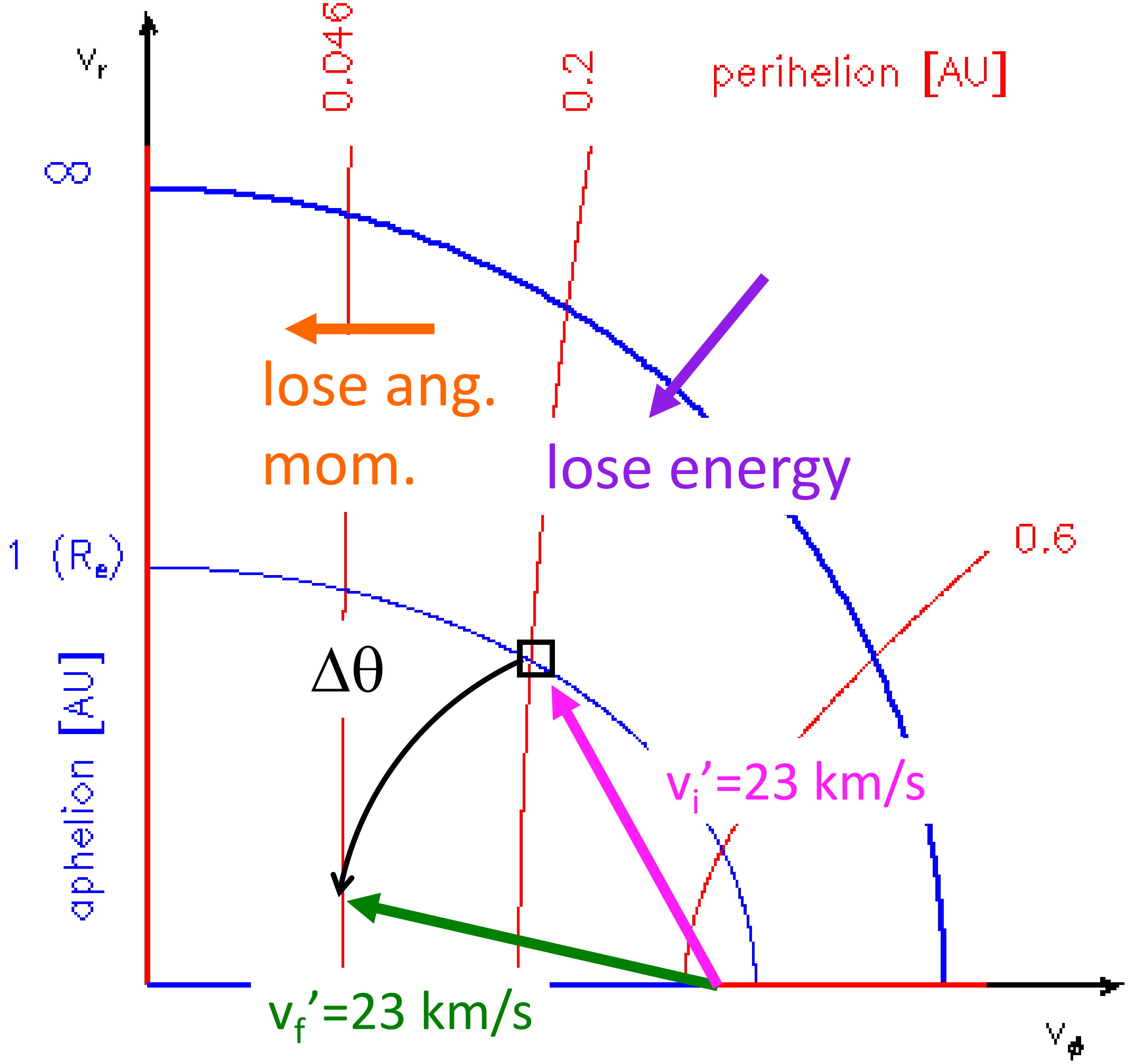


Venusian frame

elastic collision

$|v_f'| = |v_i'|$

$\Delta\theta=50^{o}$



lose ang. mom.

lose energy

v_i'=23 km/s

v_f'=23 km/s

$$\boxed{\Delta\theta = 2\sin^{-1}\left(\frac{1}{1+2v_i'^2/v_{e,p}^2}\right)}$$

Rutherford scattering angle

$v_i' = 23$ km/s

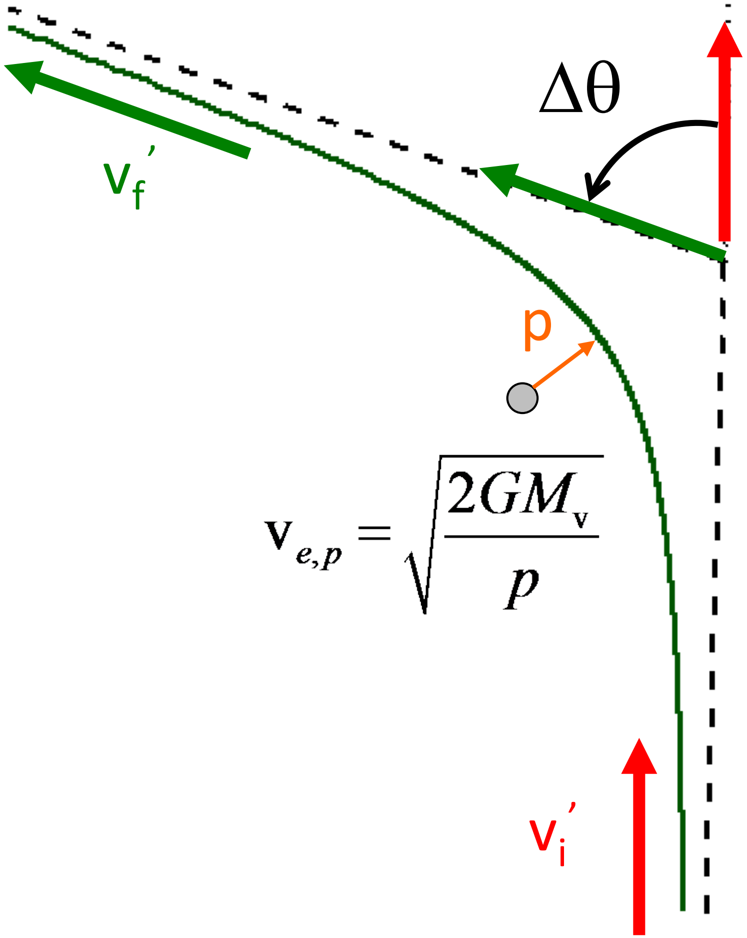
$\cancel{\Delta\theta = 50^\circ}$

→ $v_{e,p} = 28$ km/s

→ $\boxed{p = 770 \text{ km}}$

surface of Venus @

$R_v = 6{,}000$ km



$$v_{e,p} = \sqrt{\frac{2GM_v}{p}}$$

$$\boxed{\Delta\theta = 2\sin^{-1}\left(\frac{1}{1+2v_i'^2/v_{e,p}^2}\right)}$$

$v_i' = 23$ km/s

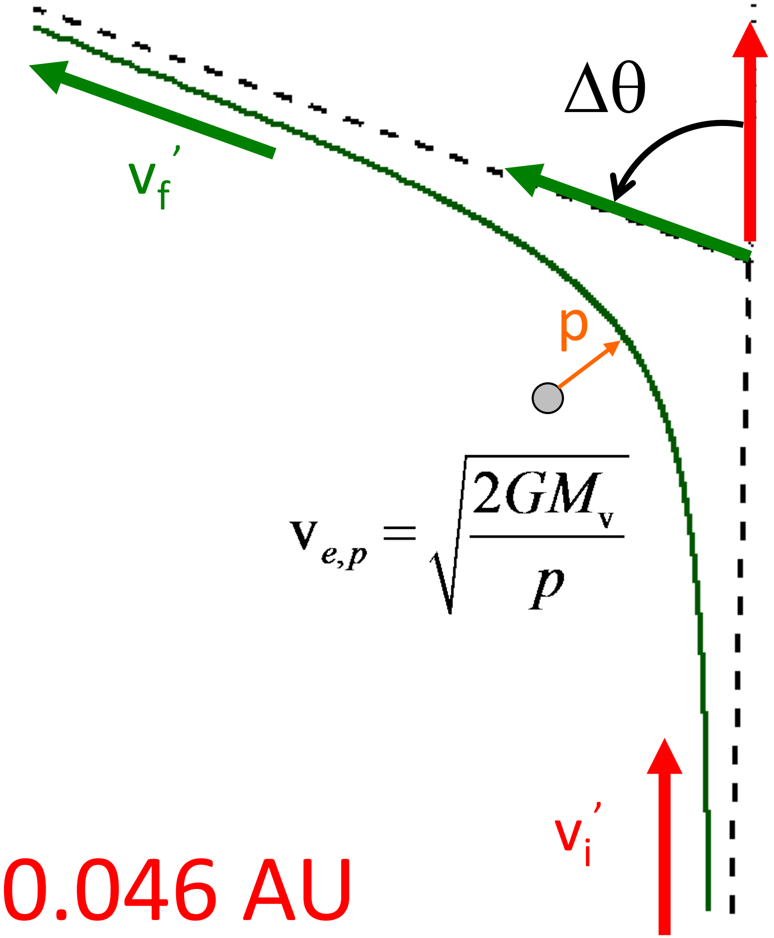
$p > R_v = 6{,}000$ km

$v_{e,p} < 10.0$ km/s

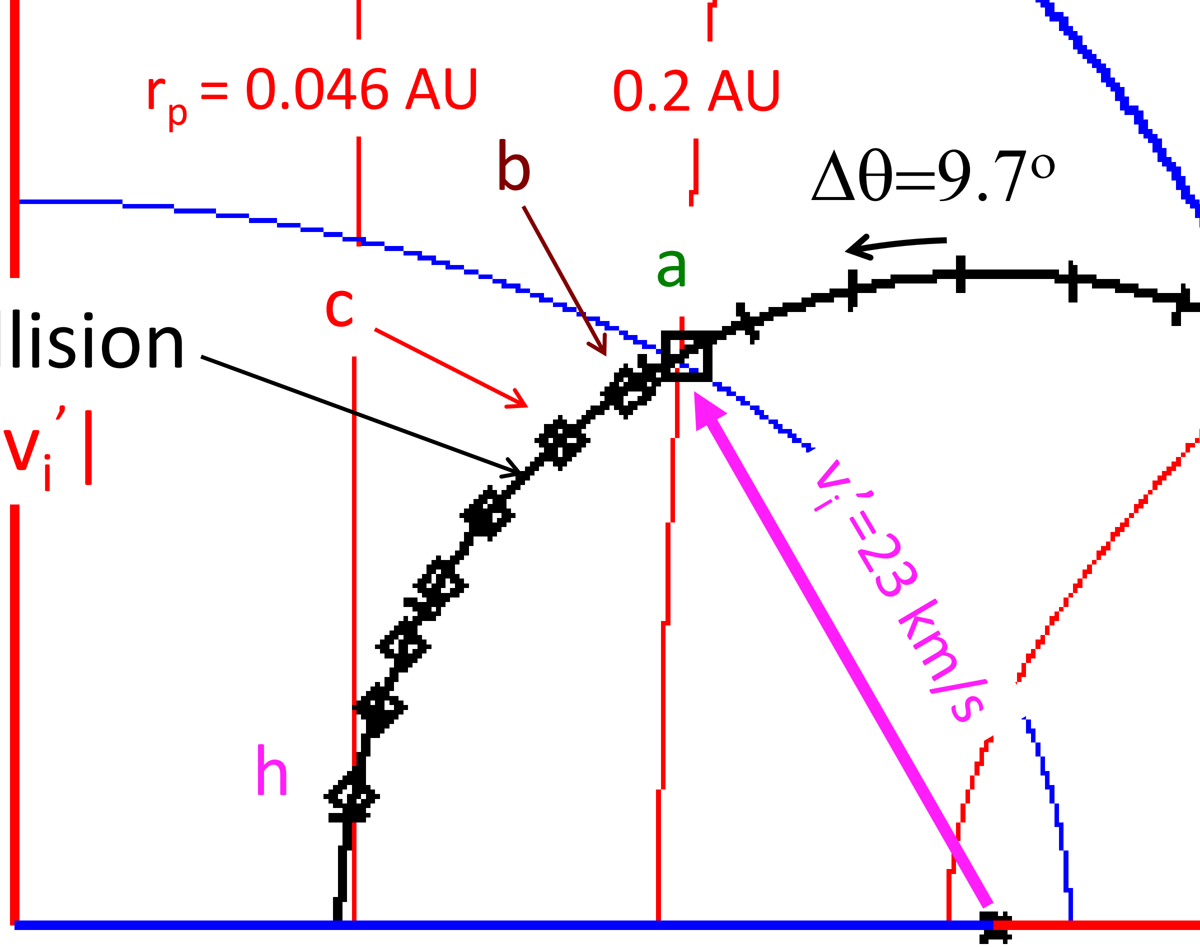
→ $\boxed{\Delta\theta < 9.7^o}$

$50^o > 5\,\Delta\theta$

→ 6+ flybys for $r_p = 0.046$ AU

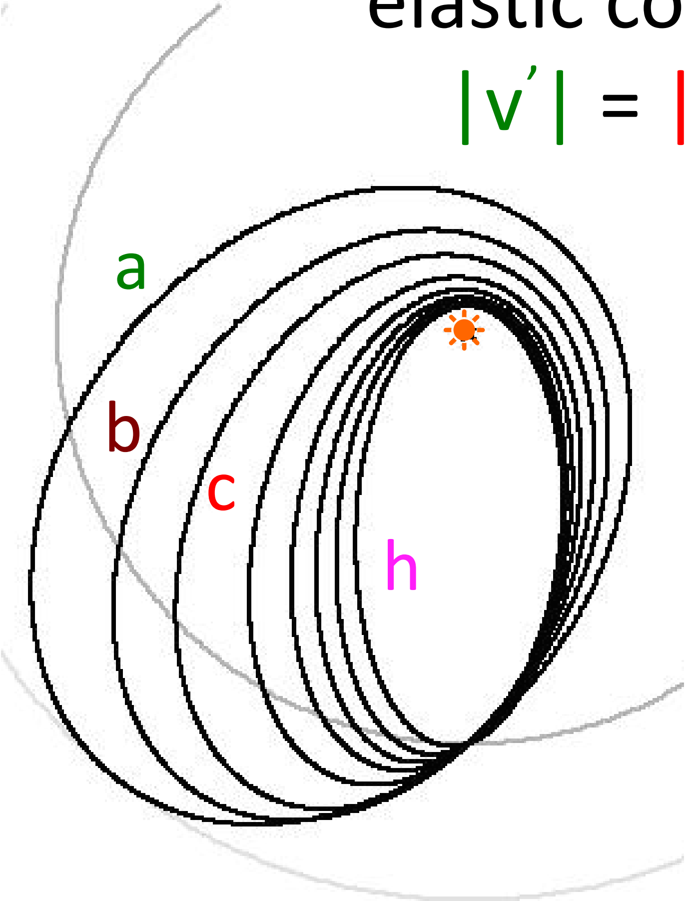


$$v_{e,p} = \sqrt{\frac{2GM_v}{p}}$$



elastic collision

$|v'| = |v_i'|$



Venus

Earth

8 orbits of PSP

➔ 7 flybys

Why not just 6? Why the crazy spacing?

Kepler 3rd: orbital period

$$P \propto a^{3/2} \propto \left(2 - \mathrm{v}^2 / \mathrm{v}_o^2\right)^{-3/2}$$

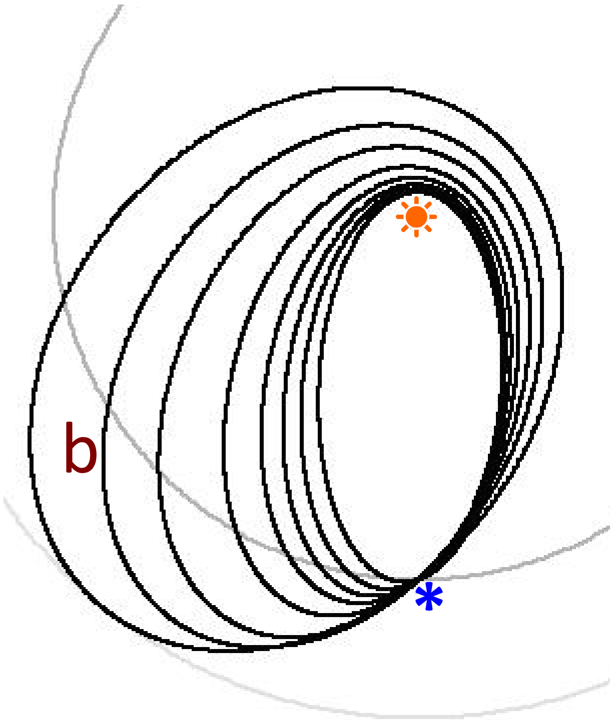
orbit **b**: P= 2/3 P_v

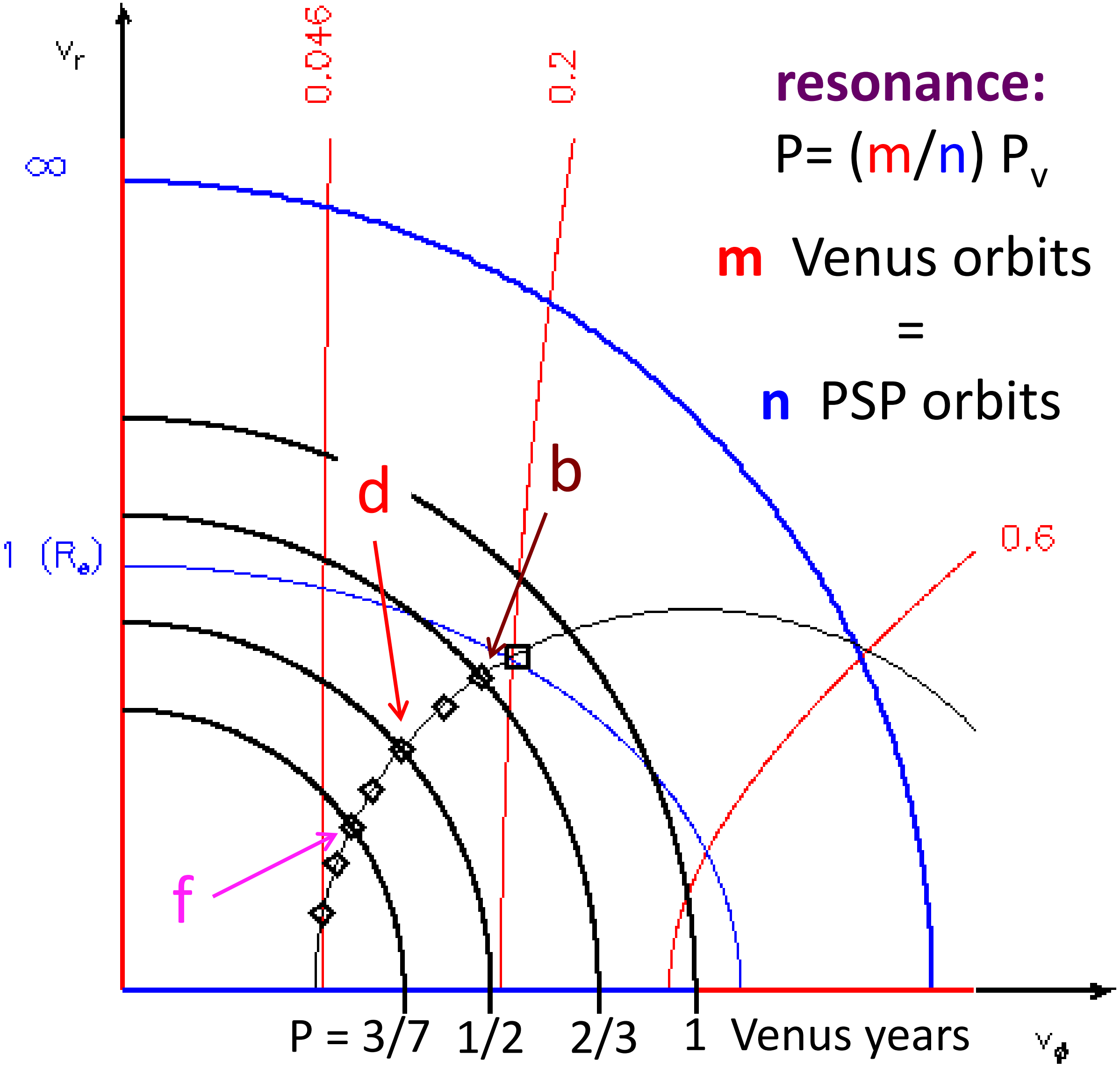
2 orbits by Venus

3 orbits by PSP

in 450 days

return to *





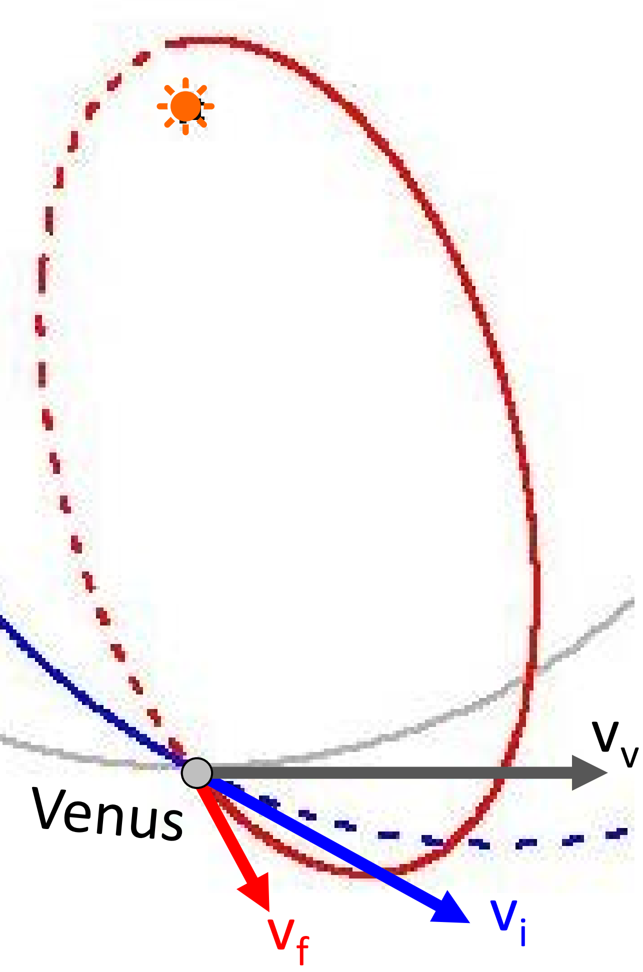
resonance:

P= (m/n) P_v

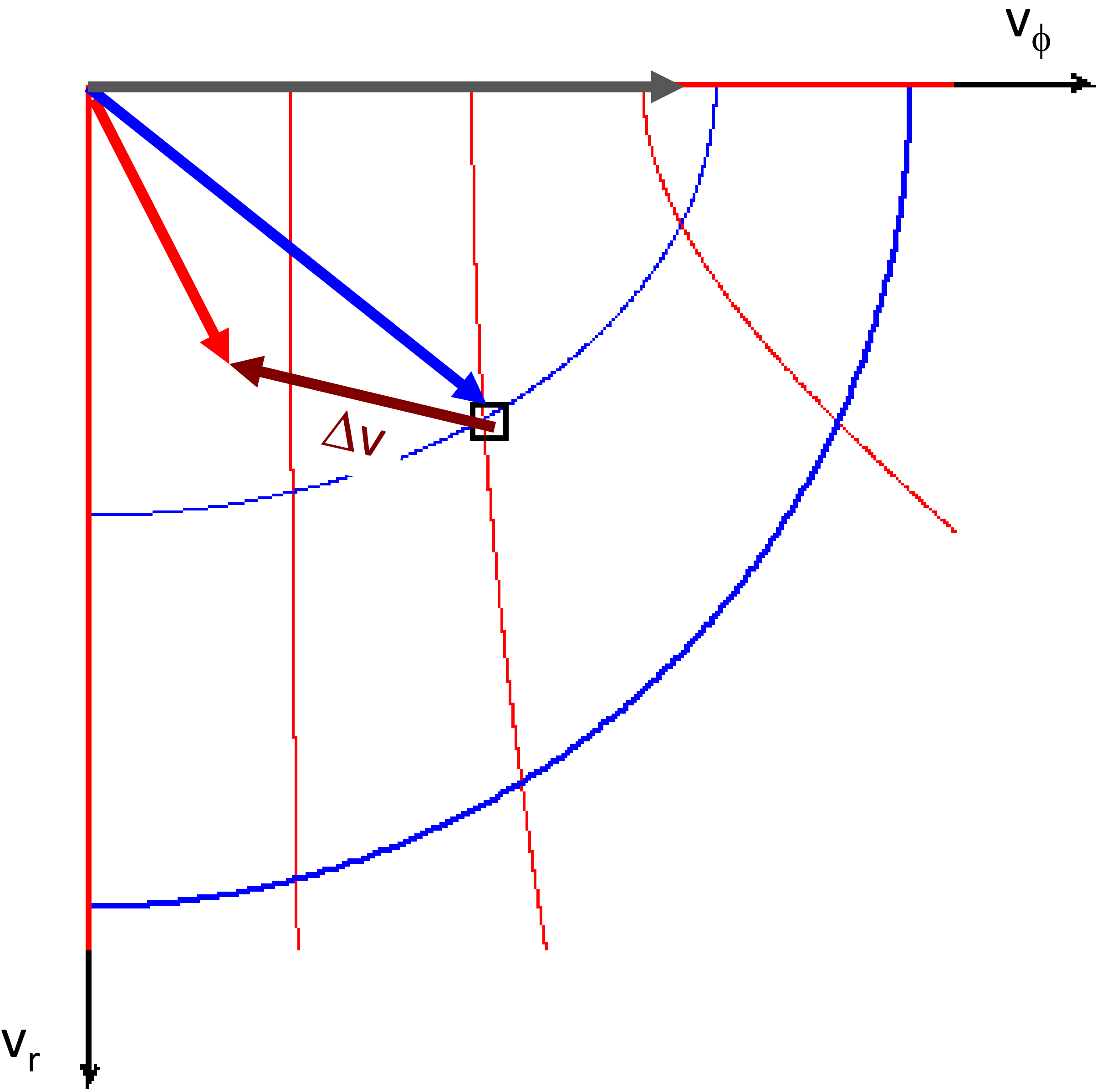
m Venus orbits

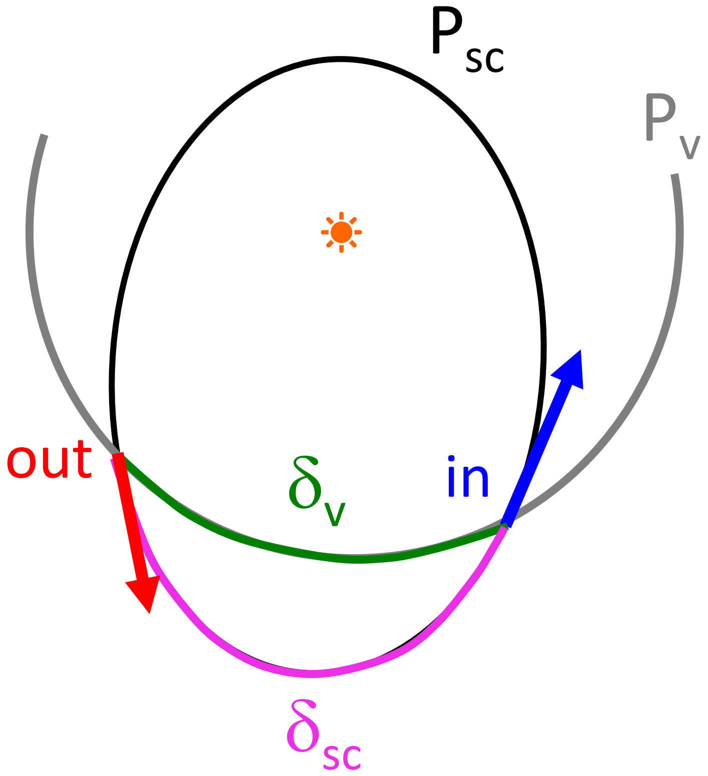
=

n PSP orbits



outbound slingshot





in-in or out-out rendezvous

➔ m : n **resonance**

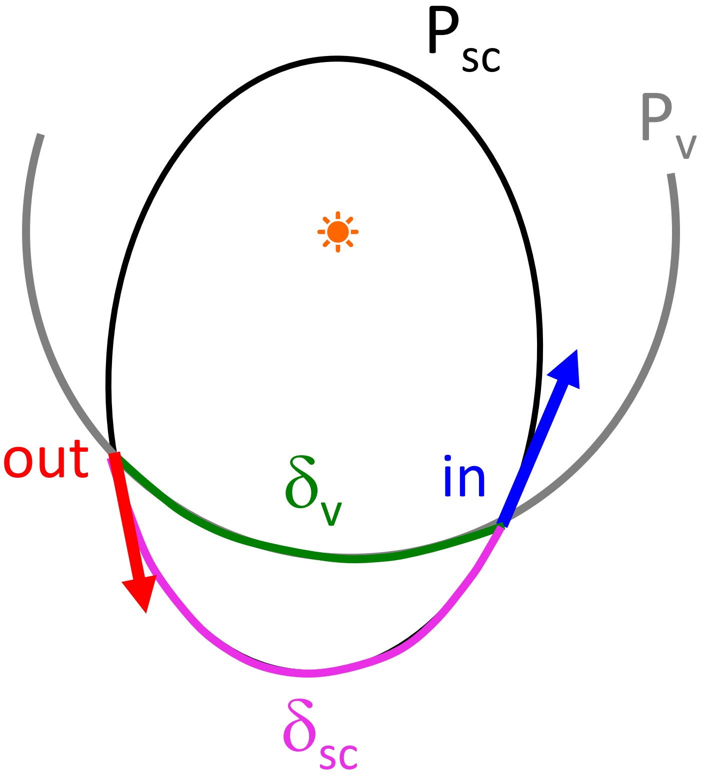
$$m P_v = n P_{sc}$$

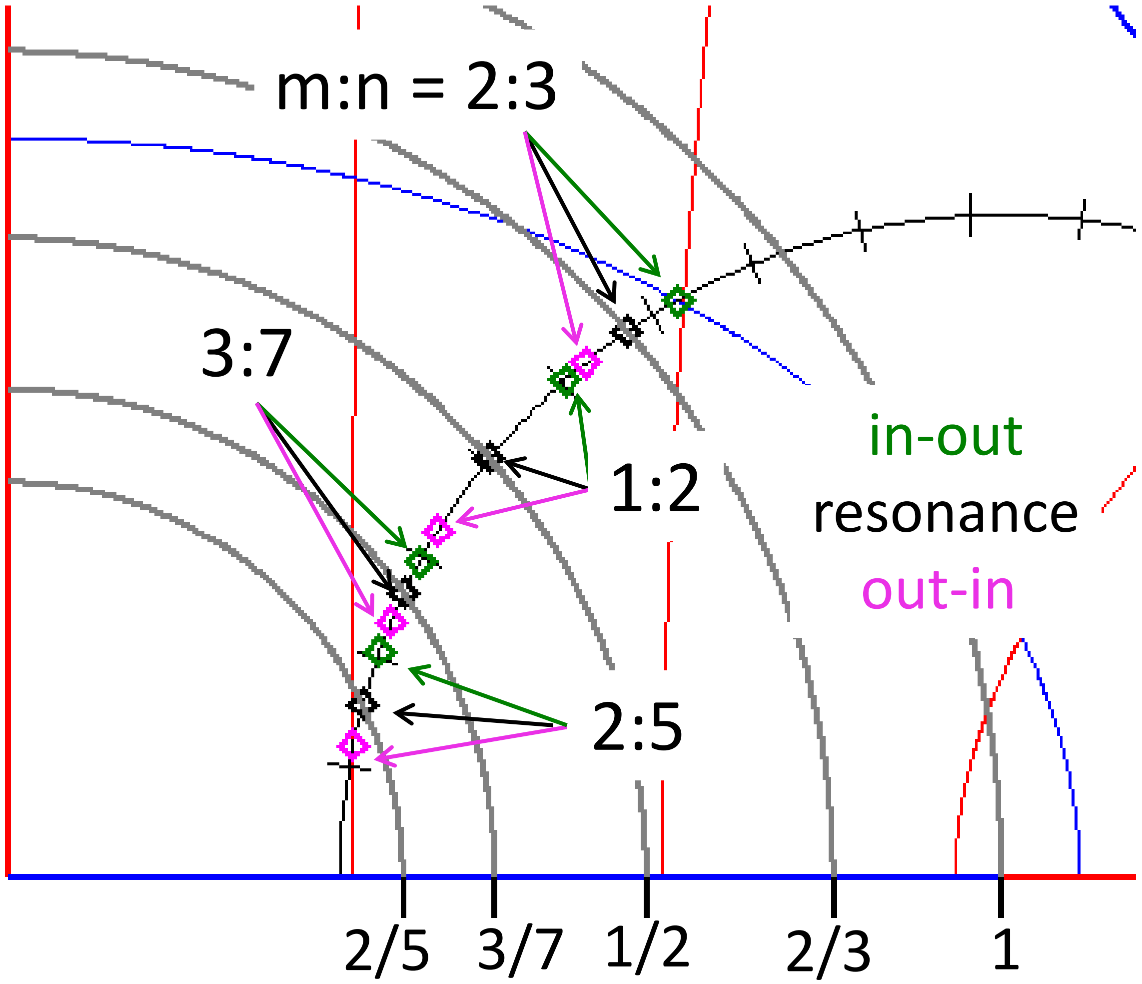
in-out rendezvous:

$$m P_v - \delta_v = n P_{sc} - \delta_{sc}$$

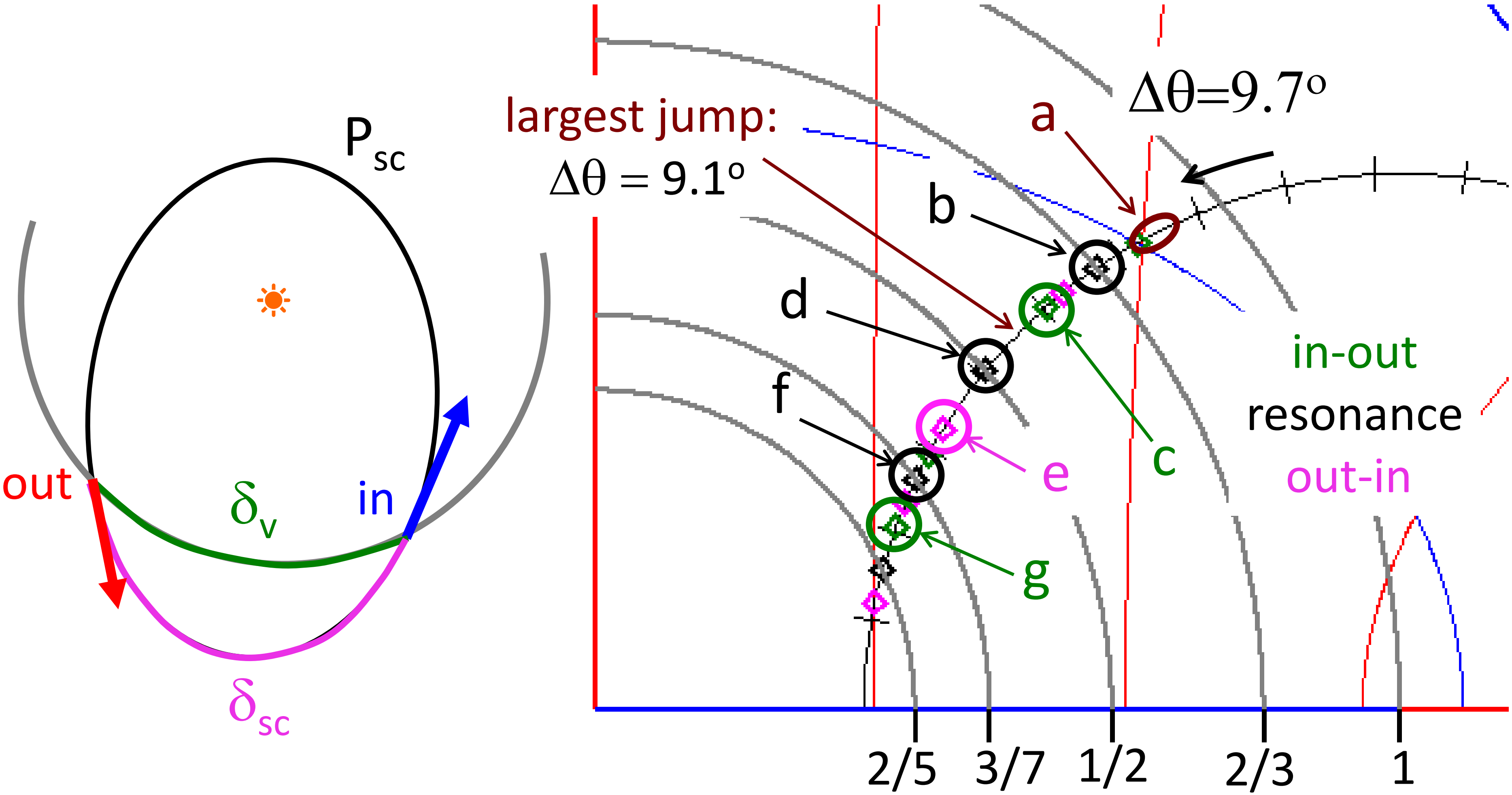
out-in rendezvous:

$$m P_v + \delta_v = n P_{sc} + \delta_{sc}$$



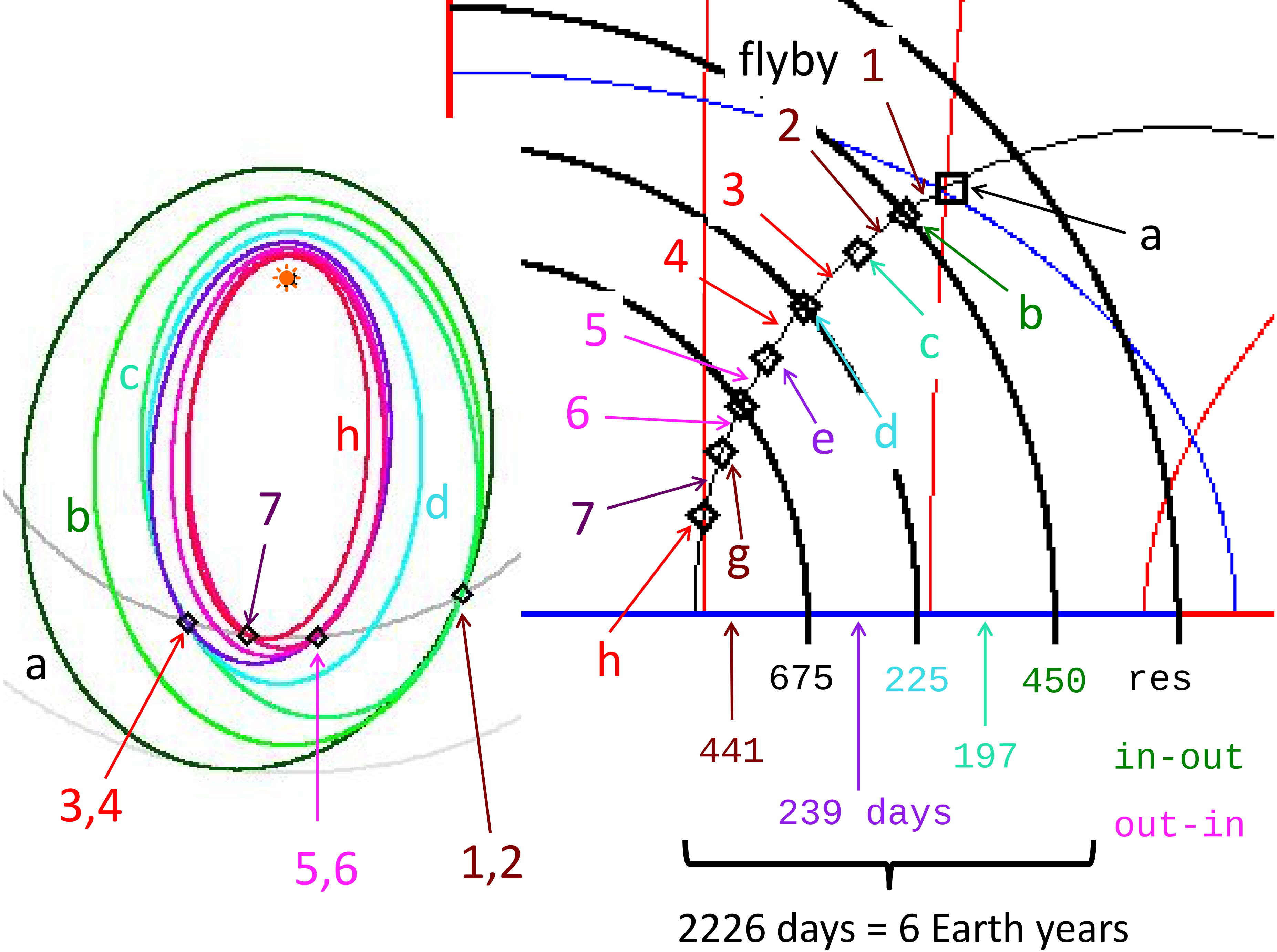


orbits allowing repeated flybys
after $\Delta t \leq 3\ P_v$

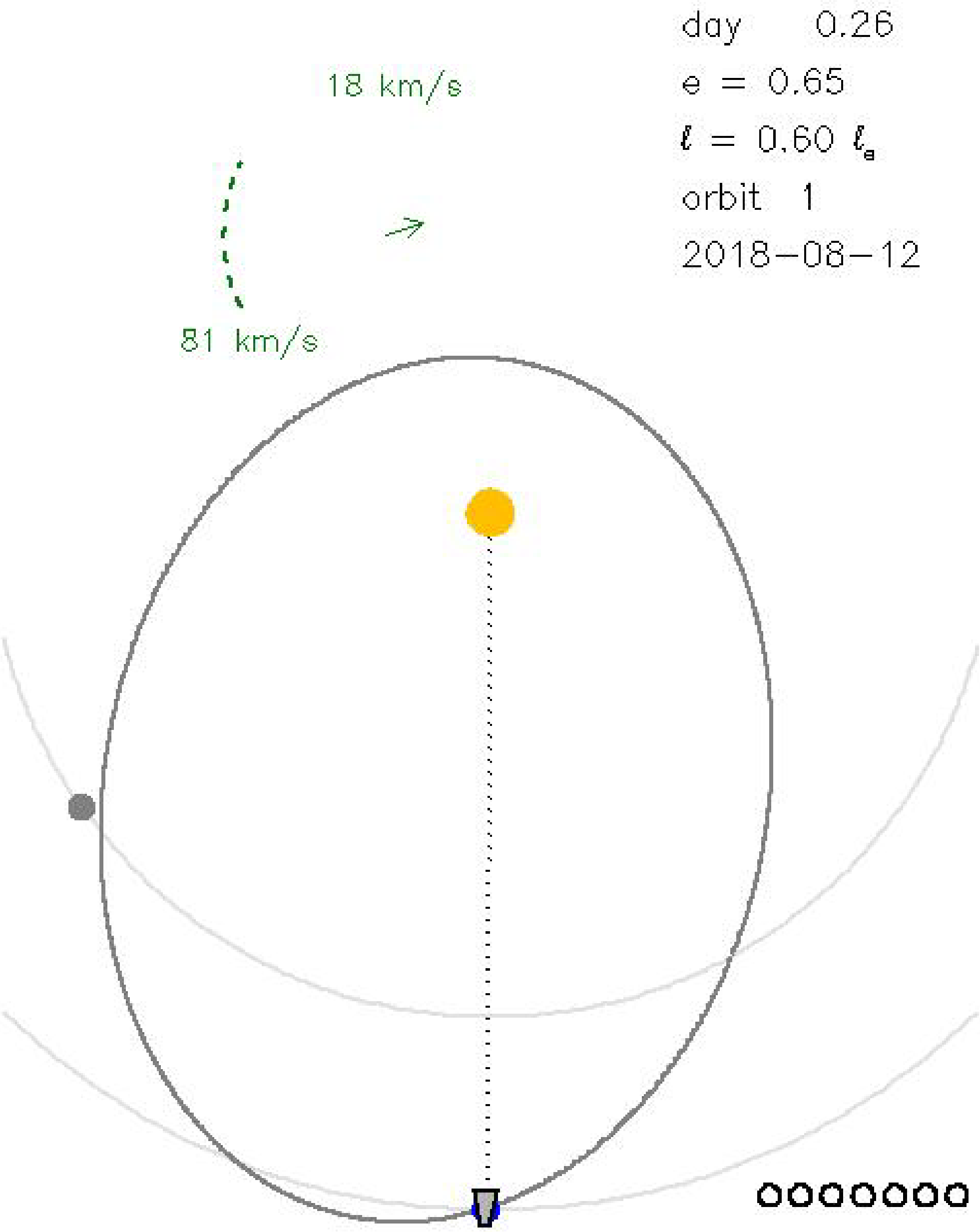


legal moves:

- in-out → out-in [via out-out (res)]
- out-in → in-out [via in-in (res)]



c
h
b
7
d
a
3,4
5,6
1,2
flyby
1
2
3
4
5
6
7
a
b
c
d
e
g
h
675
225
450
res
441
197
239 days
in-out
out-in
2226 days = 6 Earth years

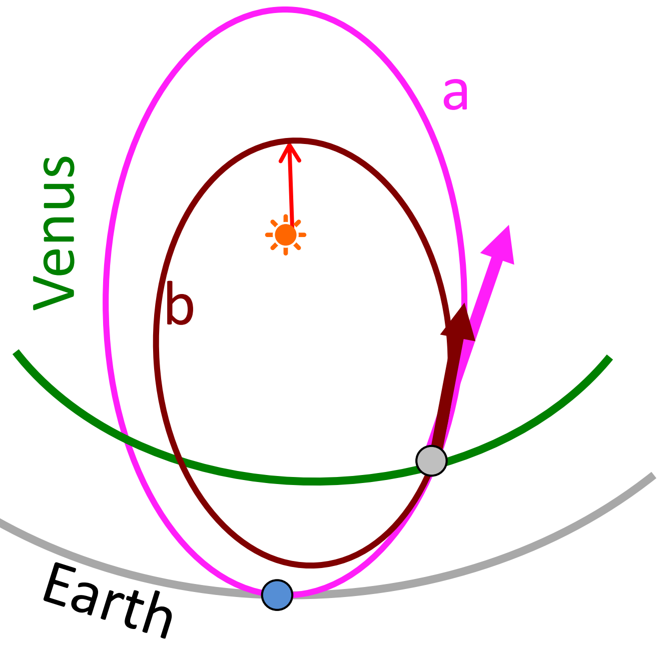

day 0.26
e = 0.65
$\ell = 0.60\ \ell_s$
orbit 1
2018-08-12
18 km/s
81 km/s
ooooooo

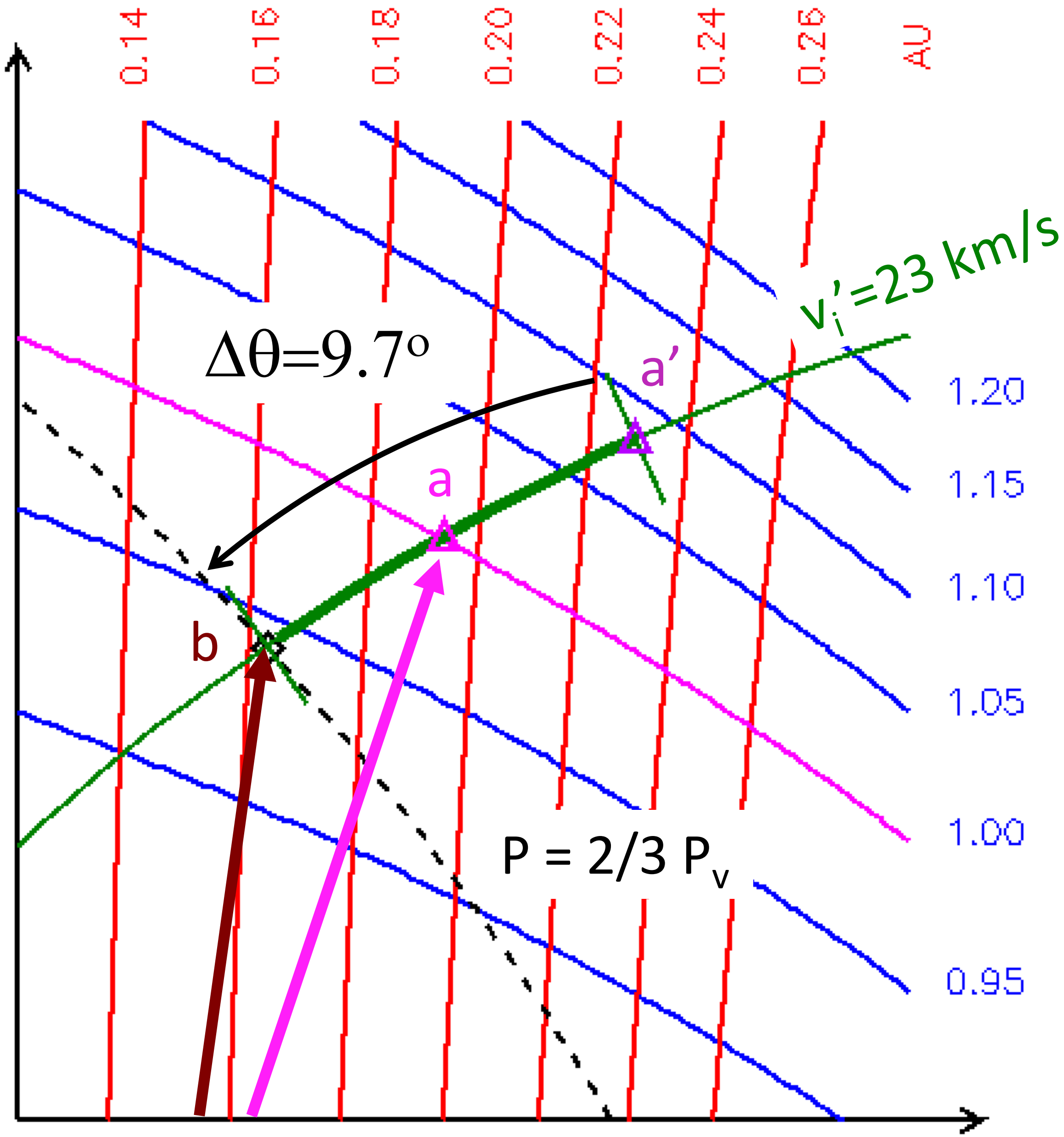
destination:

orbit b

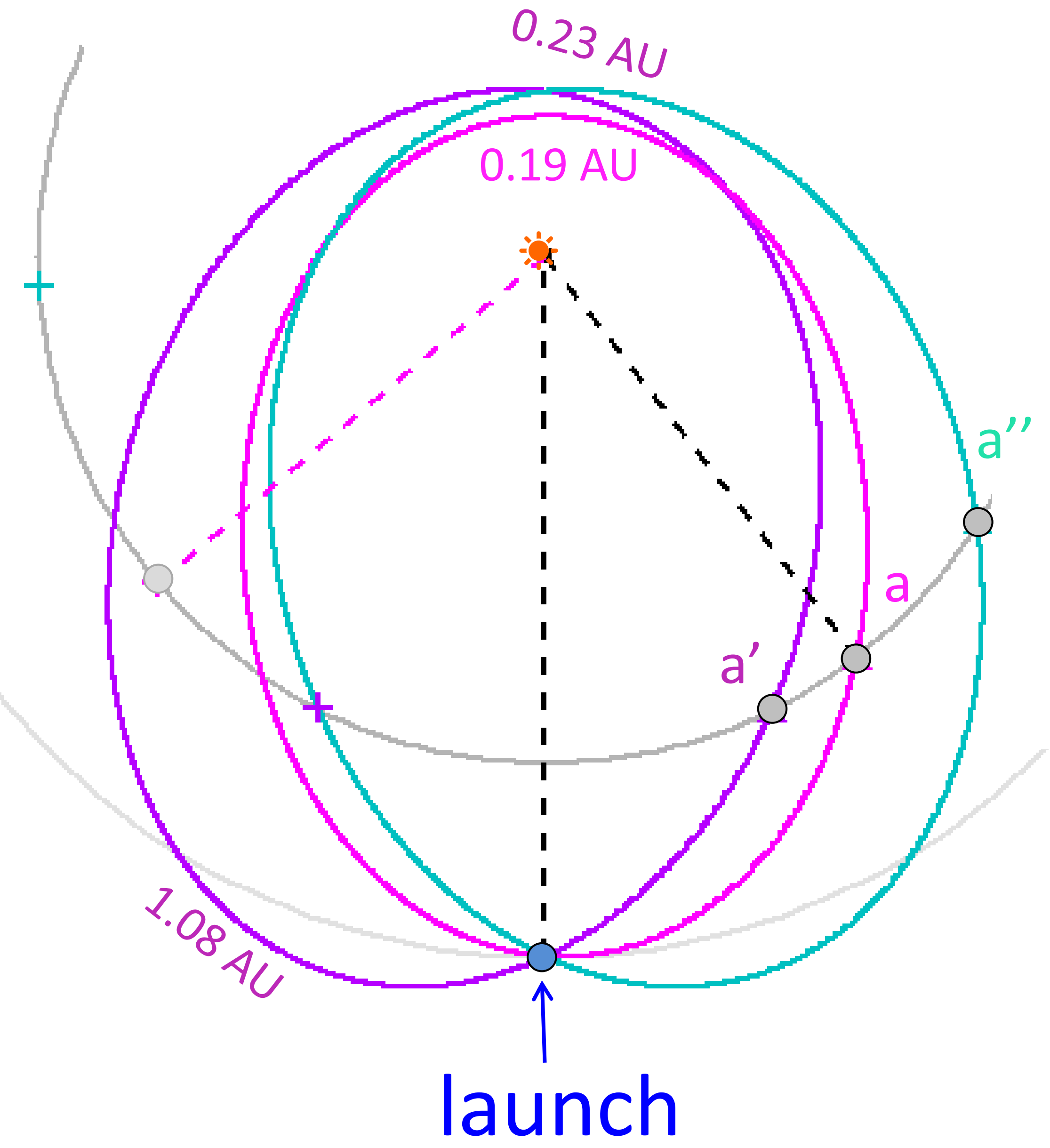
$r_a = 0.94$ AU

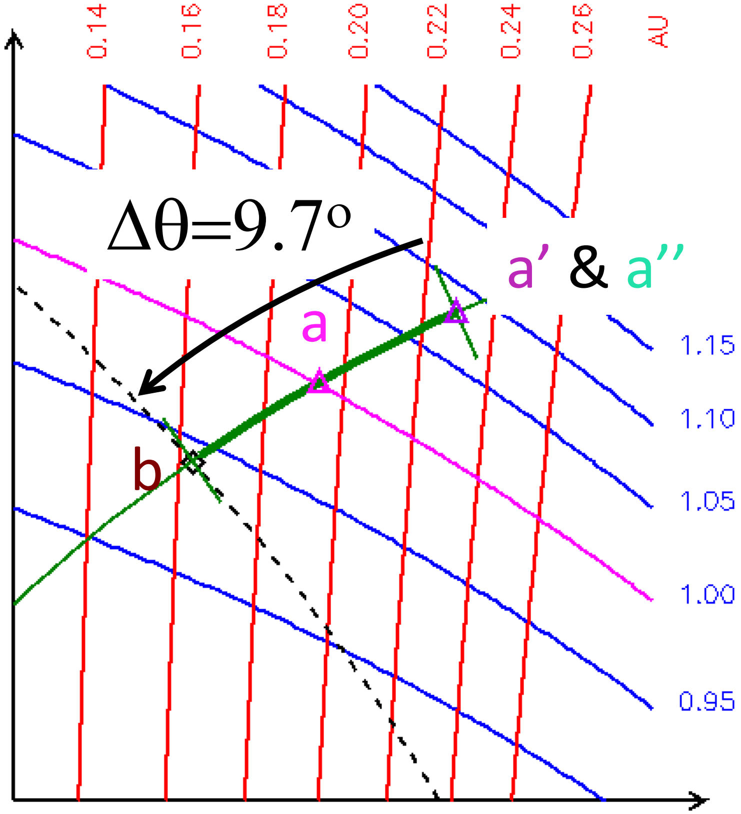
$r_p = 0.16$ AU





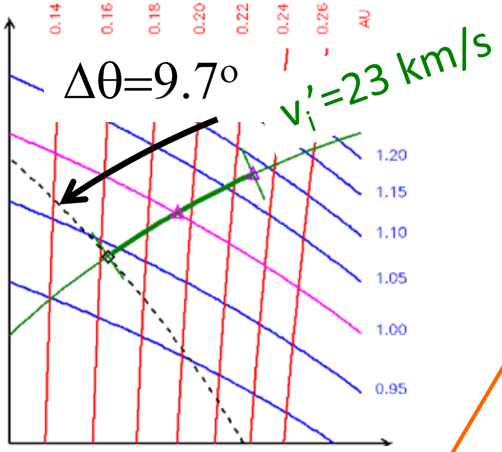
1^{st} orbit: a


0.23 AU
0.19 AU
a''
a
a'
1.08 AU
launch


0.14
0.16
0.18
0.20
0.22
0.24
0.26
AU
Δθ=9.7°
a' & a''
a
b
1.15
1.10
1.05
1.00
0.95

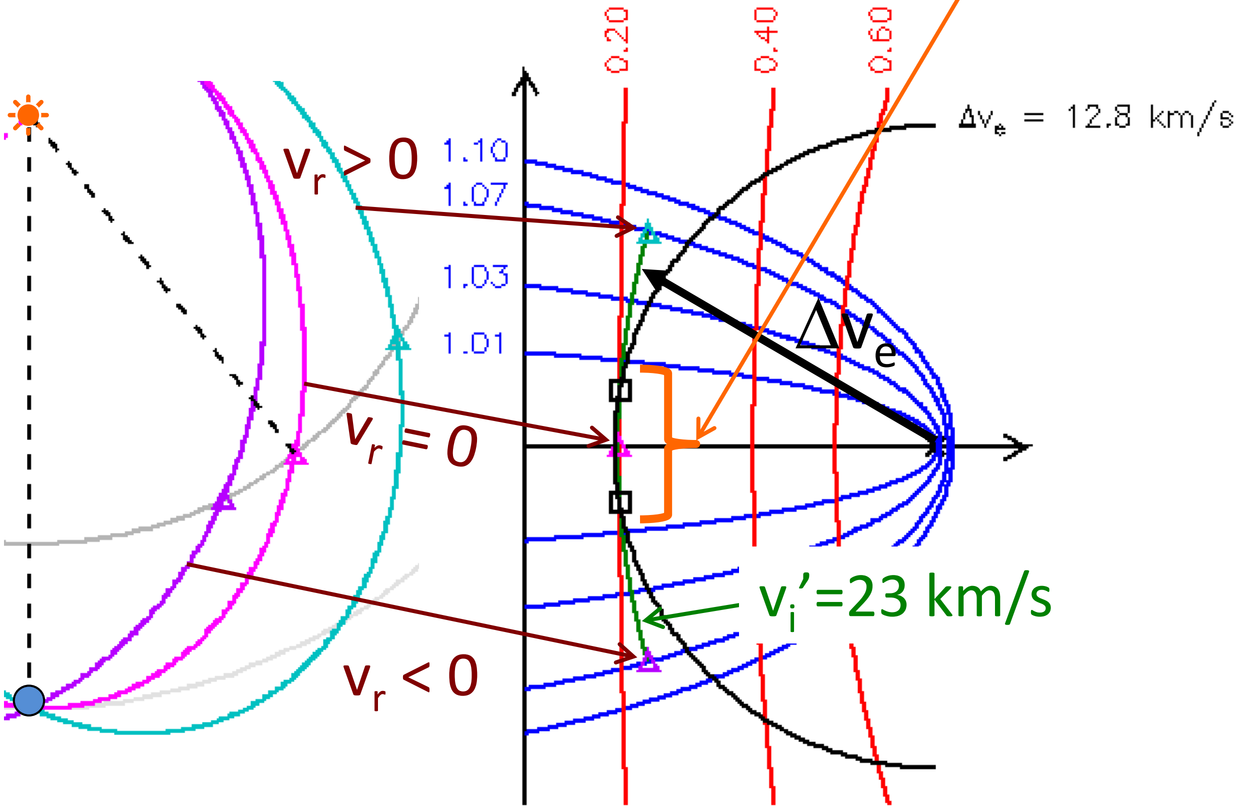
1st orbit: a

R = 0.72 AU



Launch criteria:

- $\Delta v_e < 12.8$ km/s
- $v_i' = 23$ km/s
- $\Delta\theta < 9.7^o$



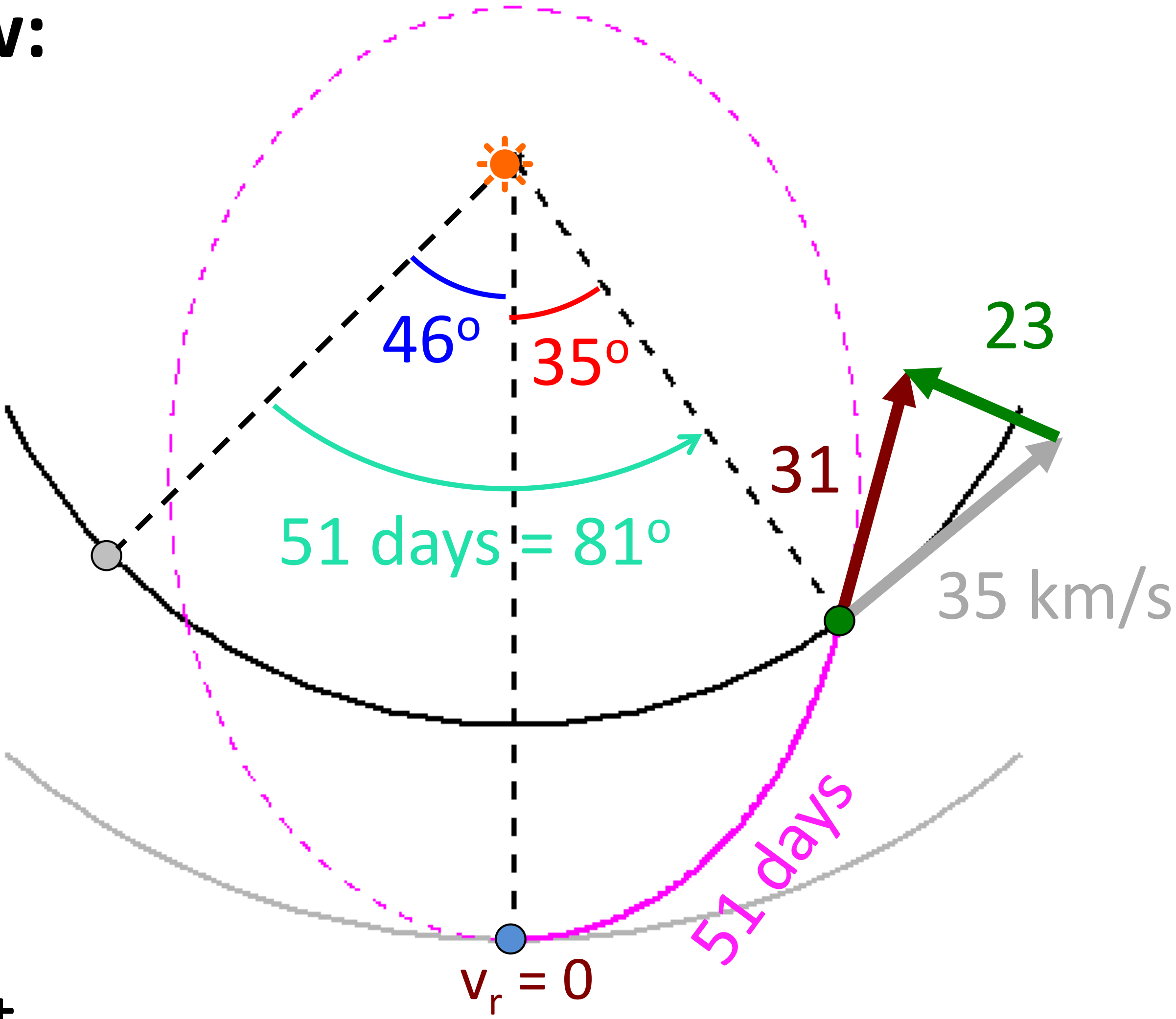
R = 1AU

Center of the Launch window:

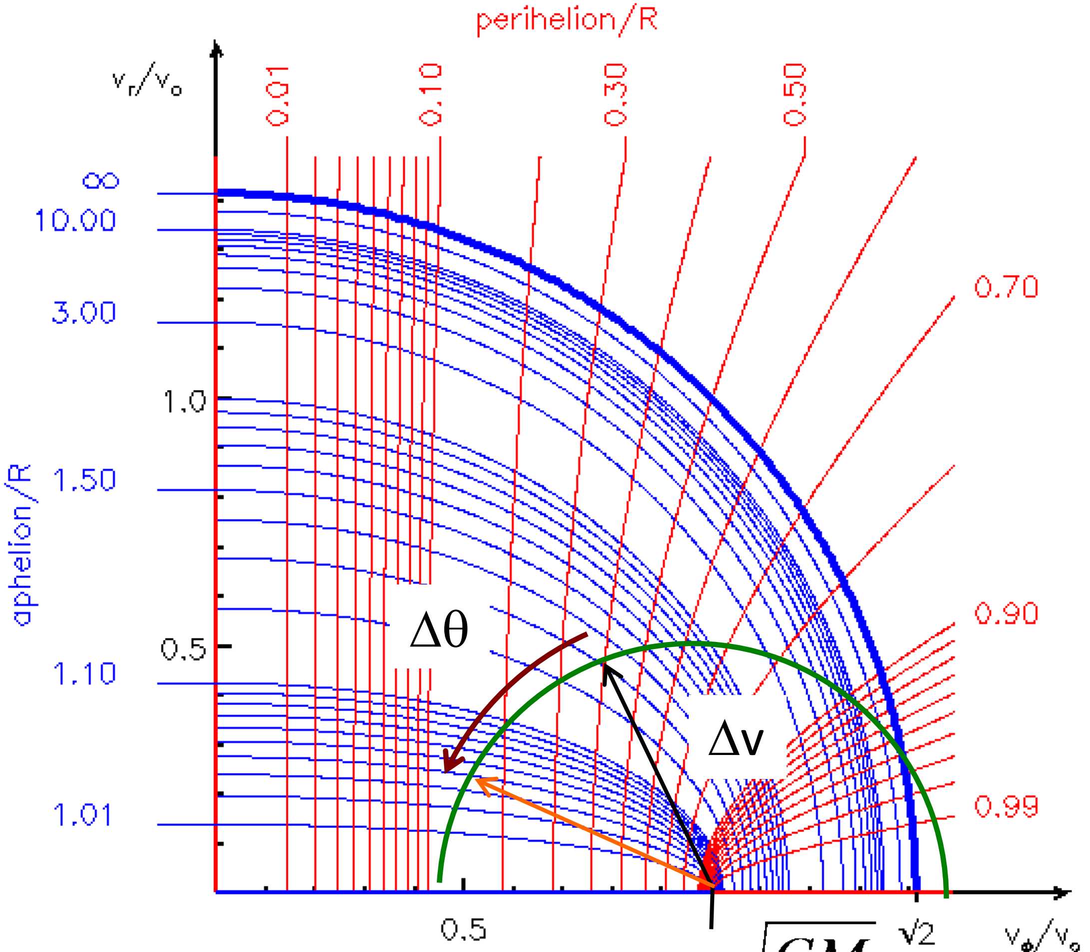
Venus lags Earth by 46° on:

- 2015-06-05
- 2017-01-09
- 2018-08-12
- 2020-03-20

19 months apart



Which Planets are Good?



$$v_o = \sqrt{\frac{GM}{R}}$$

planetary orbital speed

$$\Delta\theta = 2\sin^{-1}\left(\frac{1}{1+2v_i'^2/v_{e,p}^2}\right)$$

$$v_i' = \Delta v \sim v_o$$

characteristic scattering angle

$$\boxed{\Delta\theta_{ch} = 2\sin^{-1}\left[\frac{1}{1+2\left(v_o/v_{e,s}\right)^2}\right]}$$

surface escape speed

Which Planets are Good?

characteristic scattering angle

$$\Delta\theta_{ch} = 2\sin^{-1}\left[\frac{1}{1+2\left(\mathrm{v}_{e,s}/\mathrm{v}_{o}\right)^{-2}}\right]$$

	Mercury	Venus	Earth	Mars	Jupiter
V_o [km/s]	48	35	30	24	13
$V_{esc,s}$ [km/s]	4.2	10.4	11.2	5.0	60
$V_{esc,s} / V_o$	0.087	0.28	0.37	0.21	4.6
$\Delta\theta_{ch}$	0.4°	4.5°	7.2°	2.4°	132°

Summary

- Launch speed (rel. to Earth) Δv_e limited by rocket
- Can get extra Δv from gravity assist: scattering off moving scatterer (planet)
- Elastic scattering ➔ $|\Delta v_p|$ conserved
- Scattering angle = Rutherford scattering angle
- Limited by surface escape speed
- May need multiple scatterings (grav. assists) to achieve desired orbit
- Multiple assists require resonant orbits, in-out or out-in rendezvous orbits