Mixing efficiency in stratified turbulence

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Abstract

We investigate how the mixing efficiency in stratified turbulence is affected by the strength of the stratification. We show that the mixing coefficient $\Gamma \sim Fr^{-2}$ for weakly stratified turbulence, where $\Gamma = \epsilon_p/\epsilon_k$ and $Fr = \epsilon_k/(N u_h^2)$ is the turbulent Froude number, taken as $Fr \gg 1$ in our scaling analysis. A series of direct numerical simulations of forced turbulence with uniform stratification N confirm that $\Gamma \propto Fr^{-2}$ for $Fr > 1$. In the simulations the Froude number is then decreased below $Fr = 1$ and we find $\Gamma_{\max} = 0.51$ at $Fr \approx 0.3$ suggesting efficient mixing despite a stronger vertical stability. At even lower Fr, there is an approach towards a constant Γ of order unity in accordance with the strongly stratified turbulence theory. We briefly discuss the implications our results may have on mixing efficiency parametrizations based on the buoyancy Reynolds number.

1 Introduction

Turbulent mixing in the atmosphere and oceans is a key factor to consider when estimating global energetics. In oceanographic applications, the density perturbation ρ away from the background density profile ρ_0, and hence defined as $\rho = \rho_{\text{tot}} - \rho_0$, acts as the scalar field. The corresponding eddy diffusivity is defined as $K_\rho = B/N^2$; this is actually the eddy diffusivity for the buoyancy field $b = -\rho g/\rho_{\text{ref}}$ since it is the ratio of the buoyancy flux $B = -\langle b u_z \rangle$ to the mean background buoyancy gradient $N^2 = d\overline{b_0}/dz$. The buoyancy flux can be modelled as $B = -\langle b u_z \rangle = K_\rho \, d\overline{b_0}/dz$, analogously to how the Reynolds stresses are typically calculated. We denote volume averaging over the physical domain by $\langle \ldots \rangle$ while time averaging over the statistically steady period of a quantity is denoted by the overbar $\overline{(\ldots)}$. Note that we mostly consider stratified turbulence in the absence of a mean flow so we do not use primes for turbulence quantities and simply write them as $\mathbf{u} = (u_x, u_y, u_z)$, ρ and b.

Starting from the eddy diffusivity framework, Osborn (1980) introduced the flux Richardson number $Ri_f = B/(B+\epsilon_k)$, which is the ratio of buoyancy flux to turbulence production and can be thought of as a mixing efficiency. The mixing coefficient is similarly defined as $\Gamma = B/\epsilon_k$ related to the eddy diffusivity as $K_\rho = \Gamma \epsilon_k/N^2$. The kinetic energy dissipation rate is defined as $\epsilon_k = 2\nu \langle S_{ij} S_{ij} \rangle$ where S_{ij} is the velocity gradient tensor. A constant mixing efficiency $Ri_f = 0.17$ was assumed by Osborn (1980) leading to a mixing coefficient $\Gamma = Ri_f/(1 - Ri_f) = 0.2$, a value which has been used in oceanographic applications ever since. More recently, Salehipour and Peltier (2015) have suggested to use the potential energy dissipation rate ϵ_p instead of the buoyancy flux when calculating Γ because in steady-state stratified turbulence $B = \epsilon_p$ and the irreversible conversion of available potential energy into background potential energy due to mixing is given by ϵ_p. The potential energy dissipation rate is defined as $\epsilon_p = (\mathcal{D}/N^2)\langle \nabla b \cdot \nabla b \rangle$ since the available potential energy is $E_p = \langle b^2 \rangle/(2N^2)$ and $\mathcal{D}$ is the buoyancy diffusivity. Support for the eddy diffusivity expression as $K_\rho = \epsilon_p/N^2$ was provided by the work of Lindborg and Brethouwer (2008) who derive an analytical expression for the mean square particle displacement $1/2\langle \delta z^2 \rangle$, which increases linearly in time with a constant of proportionality equal to K_ρ. We therefore stick to these definitions of $\Gamma = \epsilon_p/\epsilon_k$ for the mixing coefficient and $Ri_f = \epsilon_p/(\epsilon_k + \epsilon_p)$ for the mixing efficiency.

In general, the mixing efficiency in stratified flows is not constant but varies in a certain parameter range. Classical parametrizations of mixing have focused on the bulk Richardson number Ri_b, which is related to the Froude number as $Ri_b \sim Fr^{-2}$. In experiments of mixing across a density interface the entrainment velocity u_e was measured and it was found that u_e/u, where u is the turbulent velocity, reaches a constant in the case of weak stratification, implying that $Ri_f \propto Ri_b(u_e/u) \propto Ri_b$ (see Turner (1973)). At the other end of the spectrum, strong stratification leads to an entrainment velocity $u_e/u \propto Ri_b^{-1}$ in the experiments by Kato and Philipps (1969). Hence a constant mixing efficiency follows, as has been confirmed by several more recent strongly stratified turbulence experiments (see Olsthoorn and Dalziel (2015) and references therein). In the research work of Shih et al. (2005) a constant mixing coefficient $\Gamma \approx 0.2$ was found for buoyancy Reynolds numbers in the range $7 < Re_b < 100$ in a series of DNS of stratified sheared turbulence but the mixing coefficient then varied as $\Gamma \propto Re_b^{-1/2}$ for $Re_b > 100$. Ocean field measurements by Davis and Moninsmith (2011) have found similar variations of $\Gamma \propto Re_b^{-1/2}$ at high $Re_b > 100$. Atmospheric boundary layer measurements by Lozovatsky and Fernando (2013) have a similar variation of Γ with Re_b, albeit at $Re_b > 10^4$ suggesting a very different bound on the buoyancy Reynolds number.

In summary, the parameters that seem to affect mixing in stratified turbulence are the Froude number and the buoyancy Reynolds number. We define these quantities and the Reynolds number based on the Taylor microscale as (with $u_h^2 = \langle u_x^2 + u_y^2 \rangle/2$),

$$Fr = \frac{\epsilon_k}{N u_h^2}, \quad Re_b = \frac{\epsilon_k}{\nu N^2}, \quad Re_\lambda = \frac{\sqrt{15} u_h^2}{\sqrt{\nu \epsilon_k}}, \tag{1}$$

from which it is clear that $Re_b = (1/15) Re_\lambda^2 Fr^2$. These definitions are different from the classical ones using a horizontal turbulent lengthscale ℓ_h. According to the strongly stratified turbulence theory developed by Billant and Chomaz (2001) and Lindborg (2006) this can be done since $\epsilon_k \sim u_h^3/\ell_h$ and $\epsilon_p \sim u_h^3/\ell_h$ as a result of equipartition of kinetic and potential energy. This brings about the prediction that the mixing coefficient $\Gamma \sim 1$ in strongly stratified turbulence.

The objective of this paper is to determine the dependent parameter affecting Γ in stratified turbulent mixing from weakly stratified turbulence to turbulence strongly affected by the stratification. Is the important physical parameter Fr or Re_b? We consider a single-parameter approach to estimating the mixing efficiency because of the idealized nature of the problem under scrutiny: homogeneous stratified turbulence in the absence of solid boundaries or mean shear, and that is statistically stationary. In addition, we neglect the influence of the Schmidt number $Sc = \nu/\mathcal{D}$, which could be important if $Sc \gg 1$ as is the case for salt-stratified water. The Schmidt number was found to have an influence on Γ in the numerical simulations by Shih et al. (2005); Salehipour and Peltier (2015) and in many other papers.

2 Scaling Analysis

The case of very small stratification N and of $Fr \gg 1$ corresponds to a physical regime that could be occurring in turbulence generated at great depths in the ocean. In the limit of weakly stratified turbulence, the horizontal and vertical length scales can be assumed to be equal as can be done for the horizontal and vertical velocity scales and the turbulence has a single integral lengthscale ℓ and velocity scale u. Under these assumptions, the equations of motion simplify to the Navier-Stokes equations, with buoyancy effects becoming negligible to leading order as shown in our previous scaling analysis of the problem (see Maffioli et al. (2016)). Here we choose an alternative but entirely consistent approach by assuming from the start that the problem is governed by passive scalar advection of buoyancy in a turbulent flow with a mean scalar gradient given by N^2. Turbulent mixing of a passive scalar presents the well-known convective-inertial range at intermediate wavenumbers k with a Monin-Obukhov spectrum for the scalar variance $E_\phi(k) \sim \langle\chi\rangle\epsilon_k^{-1/3}\, k^{-5/3}$ (see Yeung et al. (2005)). Identifying the generic scalar quantity ϕ as the buoyancy b and hence the buoyancy dissipation rate as $\langle\chi\rangle = N^2\epsilon_p$ we can write the scalar spectrum as:

$$E_b \sim N^2\,\epsilon_p\,\epsilon_k^{-1/3}\,k^{-5/3} \qquad (2)$$

We consider linearly stratified fluids with constant density gradient and therefore $N =$ const is assumed throughout. In weakly stratified turbulence we expect the buoyancy and vertical velocity fields to be well-correlated because both physical quantities should be concentrated at large scales. Moreover, the buoyancy field has no significant feedback on the velocity field so it is clear that the vertical kinetic energy spectrum will present the classical Kolmogorov spectral form $E_w(k) \sim \epsilon_k^{2/3}\,k^{-5/3}$. The co-spectrum of buoyancy and vertical velocity $E_B(k)$ corresponds to the 3-D spectrum of buoyancy flux as a function of wavenumber k. From our hypotheses it follows that,

$$E_B(k) \sim \sqrt{E_b(k)\,E_w(k)} \sim \sqrt{(N^2\,\epsilon_p\,\epsilon_k^{-1/3}\,k^{-5/3})(\epsilon_k^{2/3}\,k^{-5/3})} \sim N\,\epsilon_p^{1/2}\,\epsilon_k^{1/6}\,k^{-5/3}, \qquad (3)$$

from which we obtain the buoyancy flux at wavenumber k as,

$$\mathcal{B}(k) \sim kE_B(k) \sim N\,\epsilon_p^{1/2}\,\epsilon_k^{1/6}\,k^{-2/3}. \qquad (4)$$

The total buoyancy flux B is well approximated by the expression in (4) if we take $k \sim 1/\ell$. This is because the large-scale contribution of $\mathcal{B}(k)$ to the buoyancy flux B is the dominant one in weakly stratified turbulence. Furthermore, statistical stationarity results in $B = \epsilon_p$ and so $\epsilon_p \sim (N\,\epsilon_k^{1/6}\,k^{-2/3})^2 = N^2\,\epsilon_k^{1/3}\,k^{-4/3}$. Similarly as for the buoyancy flux, the kinetic energy can be estimated from the 3-D energy spectrum as $u^2 \sim kE(k) \sim \epsilon_k^{2/3}\,k^{-2/3}$. We finally arrive at the expression for the mixing coefficient in weakly stratified turbulence:

$$\Gamma = \frac{\epsilon_p}{\epsilon_k} \sim \frac{N^2\,\epsilon_k^{1/3}\,k^{-4/3}}{\epsilon_k} = \frac{N^2(\epsilon_k^{2/3}\,k^{-2/3})^2}{\epsilon_k^2} \sim \frac{N^2\,u^4}{\epsilon_k^2} = \left(\frac{Nu^2}{\epsilon_k}\right)^2 = Fr^{-2}. \qquad (5)$$

This result means physically that under weak stratification conditions, turbulent flows have a mixing coefficient that decreases rapidly with increasing Fr. Also, $\Gamma \propto N^2$ meaning that the mixing coefficient is linearly proportional to the background buoyancy gradient. This result is analogous to that obtained when considering Turner's experiment of a two-layer system where the mixing efficiency $Ri_f \propto Ri_b \sim Fr^{-2}$.

3 Direct numerical simulations

We have performed DNS of stratified turbulence with uniform N. The equations being solved in our pseudo-spectral code are the incompressible Navier-Stokes equations with the Boussinesq approximation. We have included a body force $\mathbf{f}$ in the momentum equation to ensure that the turbulence reaches statistical stationarity. For the simulations at high Fr, we opted for isotropic forcing while at low Fr we utilize vortical forcing concentrated in modes with $k_z = 0$. The Schmidt number in all the DNS simulations was chosen as $Sc = 1$. More details of the numerical methods can be found in Maffioli et al. (2016). As shown in Table 1, the first 5 DNS runs with increasing resolution have a successively increasing Re_λ and decreasing Fr to keep $Re_b = \text{const} \approx 10^3$. In the second set of 5 DNS runs the stratification is decreased gradually to obtain $Fr > 1$; the 1024^3 resolution ensures $Re_\lambda \approx 240$ since this value for Re_λ is a good high value above which stratified turbulence becomes independent of Re_λ (see de Bruyn Kops (2015)). The last set of 5 runs goes up to high resolution in order to properly capture the dynamics of strongly stratified turbulence with low Fr and $Re_b > 10$, a necessary condition to sustain 3-D turbulent flow (see Bartello and Tobias (2013)).

Table 1: Relevant non-dimensional parameters and type of forcing for DNS runs.

Run	$N_x = N_y$	N_z	Fr	Re_b	Re_λ	forcing
R1kF2.9	96	96	2.90	1010	42	iso
R1kF1.6	192	192	1.64	990	74	iso
R1kF0.9	384	384	0.94	980	129	iso
R1kF0.5	768	768	0.52	960	229	iso
R1kF0.3	1536	1536	0.29	990	423	iso
1024F0.7	1024	1024	0.70	2340	266	iso
1024F1.6	1024	1024	1.58	10430	250	iso
1024F3.1	1024	1024	3.10	37370	242	iso
1024F5.9	1024	1024	5.86	133430	241	iso
1024F12	1024	1024	11.97	537250	237	iso
R200F0.14	1024	1024	0.141	200	390	vort
R57F0.09	1024	1024	0.091	57	319	vort
R14F0.04	1024	512	0.044	14	324	vort
R15F0.03	2048	512	0.035	15	432	vort
R17F0.02	4096	1024	0.020	17	805	vort

4 Results

4.1 Variation of mixing coefficient at constant Re_b

In the first simulations we keep the buoyancy Reynolds number at $Re \approx 1000$ and vary the Froude number Fr. The value chosen for Re_b is well within the energetic-regime of Shih et al. (2005) and so the mixing coefficient is expected be varying as a function of Re_b and should consequently be a constant value in our five DNS runs. As shown in Figure 1 though, we find a significant variation of Γ across the runs, which is not due to variations in Re_b, held constant, but is due to a changing Fr. The values of Γ span an order of magnitude as do the values of Fr in the simulations. At $Fr = 0.29$ we have $\Gamma = 0.51$, a high value compared to the often quoted $\Gamma = 0.2$ value due to Osborn (1980). The plots

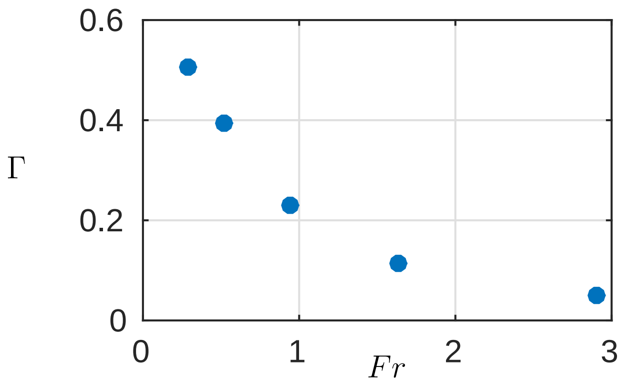

Figure 1: Mixing coefficient against Fr for the five runs from R1kF2.9 to R1kF0.3.

are made taking the time-averaged value for each DNS runs, specifically what is shown is $\Gamma = \overline{\epsilon_p}/\overline{\epsilon_k}$ and $Fr = \overline{\epsilon_k}/(N\overline{u_h^2})$, and the same procedure is applied in the next sections.

4.2 Buoyancy flux and mixing at high Fr

We turn to the DNS runs at high Fr and consider the prediction of our scaling analysis regarding the buoyancy flux spectrum. In Figure 2 the compensated form of $E_B(k)$ is plotted. The spectra are compensated according to the form given in (3) with time averaging of the quantities over the steady-state period of the forced simulations. From the plots it is clear that there is a good collapse of the DNS runs with high $Fr > 1$ for about a decade of wavenumbers, $7 \leq k \leq 70$. On the other hand, the only DNS run with $Fr < 1$, run 1024F0.7, deviates significantly from the expected form of $E_B(k)$. Interestingly, it is also the only spectrum that has a portion with $E_B(k) < 0$ at high wavenumbers, shown by the dotted line in Figure 2. This physically means that there is an exchange of potential energy back to kinetic energy at the small scales of the turbulence. We observed this phenomenon in all simulations with $Fr < 1$ and it has already been reported in other studies of strongly stratified turbulence such as in Augier et al. (2015). The good collapse of compensated $E_B(k)$ spectra gives support to the theoretical predictions of the scaling analysis. It should be noted that the full expression in (3) contains a $k^{-5/3}$ dependency of $E_B(k)$ whereas the compensated spectra shown have a limited inertial range without a clear plateau, which could be due to the moderate Re_λ of this series of DNS runs.

4.3 Mixing coefficient across the DNS dataset

We consider all runs with $Re_\lambda > 200$, for which we plot Γ as a function of Fr in Figure 3. If we focus on the high-Fr behaviour we see that $\Gamma \propto Fr^{-2}$ for $Fr > 1$, which confirms the prediction of the scaling analysis. At the other side of the plot in Figure 3 at very low $Fr < 0.04$ the mixing coefficient seems to be reaching a constant value around $\Gamma = 0.33$. This provides considerable support to the conjecture that in strongly stratified turbulence $\Gamma = \mathcal{O}(1)$. All in all, the results show that the physical parameter affecting mixing coefficient and mixing efficiency is the Froude number Fr.

5 Discussion and conclusions

We have presented results from direct numerical simulations of constant-N forced stratified turbulence covering almost 3 orders of magnitude in Fr and a big range of Re_b. The

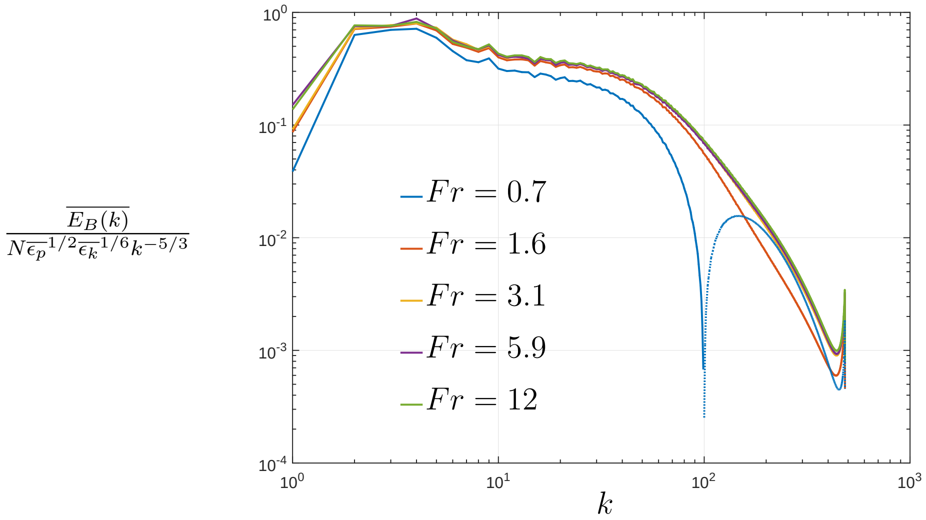

Figure 2: Buoyancy flux spectra in compensated form for the five DNS runs from 1024F0.7 to 1024F12.

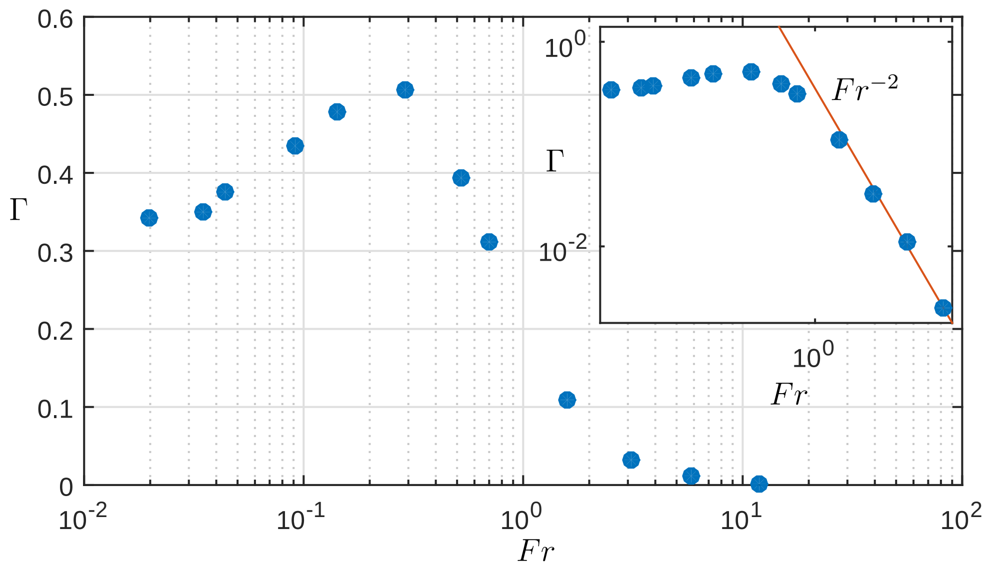

Figure 3: Mixing coefficient as a function of Froude number (log-log plot shown in inset).

simulations at high Re_λ show a non-monotonic behaviour of the mixing coefficient as a function of Fr with a peak in Γ at $Fr \approx 0.3$ where $\Gamma = 0.51$. This value is significantly larger than the $\Gamma = 0.2$ value that is commonly used in oceanographic applications. The mixing coefficient then drops to values around $\Gamma = 0.33$, which are still high values due to the presence of a strong density differences at low Fr, which high-Re_b turbulence is able to mix efficiently. The results show that using $\Gamma =$ const may not be an accurate approach for estimating mixing rates and that it is important to consider local properties of ocean turbulence before deciding on a suitable value for Γ to use. Our results point towards the Froude number as the main physical parameter influencing Γ in homogeneous stratified turbulence. This could provide a new way of interpreting recent parametrizations and field measurements of ocean mixing using the buoyancy Reynolds number.

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