A Space-Time Approach to Wave Turbulence

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The concepts of wave turbulence theory provide a fairly general framework for studying the statistical steady states in a large class of weakly interacting and weakly nonlinear many body or many wave systems driven far from equilibrium by the presence of external forcing and damping. In its essence, classical wave turbulence theory [1] is a perturbation expansion in the amplitude of the nonlinearity, supposing linear plane wave solutions of $ae^{i(\mathbf{p}\cdot\mathbf{r}-\sigma t)}$ at leading order and a slow amplitude modulation at the next order of the expansion, $a(\tau)e^{i(\mathbf{p}\cdot\mathbf{r}-\sigma t)}$, by near-resonant interactions. This modulation leads to a redistribution of the spectral energy density among space and time scales.

The holy grail of Wave Turbulence is an estimate of energy/action transport to smaller/larger scales [1] in an analogy to the 3-D cascade of [2], with the supposition that the end point of such cascades feeds 3-D turbulence in a spatially inhomogeneous temporally intermittent wave breaking process. The emphasis on the spectral domain makes sense in the context of an inertial cascade in a spatially homogeneous system, obtained after averaging over that space-time variability. But that spatial variability is *not* part of the $a(\tau)e^{i(\mathbf{p}\cdot\mathbf{r}-\sigma t)}$ description in the previous paragraph.

Significant geophysical wave systems (internal waves, Rossby waves) are non-local in that integrals over wavenumber within the kinetic equation do not close unless a lower bound is inserted. Such systems are dominated by extreme scale separated interactions. This is a domain in which ray-tracing, the wave system's particle analogy, provides an internally consistent theoretical paradigm. Ray tracing is a vehicle that admits to spatial variability.

A generic landing place for extreme scale separated Hamiltonian systems is diffusion in the spectral domain. For internal waves, wave turbulence results in an equation for the diffusion of action in vertical wavenumber,

$$\frac{\partial n(\mathbf{p})}{\partial t} - \partial_m D(\mathbf{p})\partial_m n(\mathbf{p}) = 0 \qquad [1]$$

with diffusivity

$$D((\mathbf{p}) = m^2 k_h m_* e_o^{gm}/N \; .$$

But application of this knowledge to fluids relies upon analogies [1, 3, 4] that cast waves as coupled oscillators. This neglects space-time variations associated with wave packets, for which

$$a(\mathbf{R},\tau)e^{i(\mathbf{p}\cdot\mathbf{r}-\sigma t)} \; .$$

with $\mathbf{p}$ and σ connected by a dispersion relation. This approach is limited by the lack of a systematic theory for assessing spectral transports. While ray tracing is not limited by an *explicit* assumption of weak nonlinearity, it is unclear what information has been discarded by wave turbulence.

While the emphasis on the spectral domain makes sense in the context of an inertial cascade in a spatially homogeneous system, it does not begin to speak to the space-time localization of the particle analogy, wave packets.

Here we present a ray tracing approach to wave turbulence that demonstrates the diffusive coupled oscillator paradigm can not be recovered. Rather, the small amplitude limit of the particle paradigm is a forced oscillator equation for the path.

Ray tracing revolves around an equation for action conservation,

$$\frac{\partial n_{\mathbf{p},\mathbf{R}}}{\partial t} + \nabla_{\mathbf{p}}\sigma_{\mathbf{p},\mathbf{R}}\cdot\nabla_{\mathbf{R}} n_{\mathbf{p},\mathbf{R}} - \nabla_{\mathbf{R}}\sigma_{\mathbf{p},\mathbf{R}}\cdot\nabla_{\mathbf{p}} n_{\mathbf{p},\mathbf{R}} = 0, \qquad [2]$$

or alternately

$$\frac{\partial n_{\mathbf{p},\mathbf{R}}}{\partial t} + \nabla_{\mathbf{R}}\cdot\left[n_{\mathbf{p},\mathbf{R}}\nabla_{\mathbf{p}}\sigma_{\mathbf{p},\mathbf{R}}\right] - \nabla_{\mathbf{p}}\cdot\left[n_{\mathbf{p},\mathbf{R}}\nabla_{\mathbf{R}}\sigma_{\mathbf{p},\mathbf{R}}\right] = 0. \qquad [3]$$

The characteristics of (2) are defined by

$$\dot{\mathbf{R}}(t) = \nabla_{\mathbf{p}}\sigma_{\mathbf{p},\mathbf{R}} = \mathbf{C_g} + \overline{\mathbf{u}}, \qquad \dot{\mathbf{p}}(t) = -\nabla_{\mathbf{R}}\sigma_{\mathbf{p},\mathbf{R}}, \qquad [4]$$

Consistent with the slowly varying approximation and assuming constant stratification, the internal wave dispersion relation becomes

$$\frac{\omega^2 - f^2}{N^2} = \frac{k_h^2}{m^2} \; . \qquad [5]$$

with intrinsic frequency $\omega(\mathbf{p}) = \sigma - \mathbf{p}\cdot\overline{\mathbf{u}}$

the coupled oscillator diffusion equation

In the spatial domain, [7] provides the following *identity* based upon tracking a Lagrangian parcel with coordinate 'x':

$$\frac{d}{dt}[x(t) - x(t=0)]^2 = 2\int_0^t \dot{x}(t)\dot{x}(t-\tau)d\tau \; . \qquad [6]$$

If the integral converges, one has diffusion with diffusivity $\mathcal{D} = \int_0^\infty \dot{x}(t)\dot{x}(t-\tau)d\tau$.

It is a simple step to substitute $\mathbf{p}$ for x and obtain a quantitative approach for discussing the migration and dispersion of wave packets in the spectral domain. Thus,

$$\frac{d^g}{dt}[m - m(t=0)]^2 = 2k^2\int_0^t U_z(t)U_z(t-\tau)d\tau \; . \qquad [7]$$

Our investigations are limited to representing the background as a sum of inertial oscillations:

$$\overline{u} = \sum_i \overline{u}_i \ \sin(\mu_i z - ft + \phi_i) \ . \qquad [8]$$

II wave number

A closed equation for m can be derived by understanding that $\dot{z} = -\frac{kN}{m^2} sgn(m)$:

$$\dot{m} = -kU_z = \Sigma_i k\mu_i cos(\mu_i \int_{-\infty}^{t} \frac{kN}{m^2} \text{sgn}(m)dt' - ft + \phi_i) \ . \quad [9]$$

However, manipulation of this does not provide much insight. See Section 5 of [5] for an indeterminate discussion.

III path

On the other hand, deriving an equation in the path proves much more useful:

$$\begin{aligned} \dot{z} &= -\frac{kN}{m^2} sgn(m) \\ \ddot{z} &= \frac{2}{\sqrt{kN}} \dot{z}^{3/2} \dot{m} sgn(m) \end{aligned} \qquad [10]$$

If we say $z = C_g^o t + z'$, and $\dot{z}' \ll C_g^o$ we arrive at a forced nonlinear oscillator equation:

$$\ddot{z}' + 2\sqrt{\frac{k}{N}} C_g^{3/2} A(t) sin(\mu_i z') = -2\sqrt{\frac{k}{N}} C_g^{3/2} B(t) cos(\mu_i z') \qquad [11]$$

$$A(t) = \Sigma_i \mu_i U_i \text{sin}[(\mu_i C_g - f)t + \phi_i]$$

$$B(t) = \Sigma_i \mu_i U_i \text{cos}[(\mu_i C_g - f)t + \phi_i]$$

If $\mu_i z' \ll 1$, we have a standard forced oscillator equation:

$$\ddot{z}' + 2\sqrt{\frac{k}{N}} C_g^{3/2} A(t) \mu_i z' = -2\sqrt{\frac{k}{N}} C_g^{3/2} B(t) \qquad [12]$$

To Do

We will discuss the implications of this and the conditions under which (12) can reduce to a diffusive approximation.

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