The sensitivity of stratified flow stability to base flow modifications

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Abstract

We present a novel theory that determines the sensitivity of linear stability to changes in the density or velocity of a base flow. The sensitivity is based on global direct and adjoint eigenmodes of the linearized Boussinesq equation, and is inspired by constant-density sensitivity analysis. The theory can be applied broadly to incompressible flows with small density variations, but it specifically provides new insight into the stability of density-stratified flows. Examples are given for the flows around a transverse thin plate at a Reynolds number of 30, a Prandtl number of 7.19, and Froude numbers of ∞ and 1. In the unstratified flow, the sensitivity is largest in the recirculation bubble; the stratified flow, however, exhibits high sensitivity in regions immediately upstream and downstream of the bluff body.

1 Introduction

In recent years, the study of global flow stability has seen a drastic shift with the introduction of sensitivity analyses. The advent of sensitivity provided computationally efficient *a priori* techniques to determine—both qualitatively and quantitatively—how particular factors would change the flow stability. These stability effects are furthermore innately related to passive and active flow control, which has great importance in the engineering of fluid systems.

Marquet et al. (2008) developed a particular sensitivity theory that explores how changes in the base flow velocity field would affect the stability of a constant-density flow. The sensitivity was then computed for a 2-D flow around a cylinder at the critical Reynolds number of $Re = 46.8$, corresponding to the onset of von Kármán vortex shedding. The computations revealed that the recirculation region downstream of the cylinder exhibited the greatest sensitivity.

In this manuscript, we extend Marquet et al.'s sensitivity to the Boussinesq equation. The theory we develop can be applied generically to flows with small density variations. Nevertheless, a special subclass of results can be obtained specifically for flows with background density stratifications. The focus on stratified flow examples is further motivated by previous studies in stratified vortex stability, including ones focusing on vortex rings (Johari and Fang, 1997), vortex pairs (Ortiz et al., 2015), pancake vortices (Billant and Chomaz, 2000), and even entire turbulent wakes (Gourlay et al., 2001).

This manuscript is structured as follows. Section 2 derives the sensitivities to base flow density and velocity modifications. Section 3 then briefly summarizes the numerical meth-

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ods that are used to compute the examples in Section 4, where we investigate the sensitivities of unstratified and stratified flows around a transverse thin plate. Finally, we summarize the results and provide concluding remarks in Section 5.

2 Theory

In this section, we develop the sensitivities to base flow density and velocity modifications, using the Boussinesq approximation. Here, we assume that the background density field is linearly stratified, and express the equations in terms of a constant Froude number Fr. The governing equations could be written more generically, however, if the density is not necessarily stratified.

For the nondimensional displacement $\mathbf{x}$ and time t, we consider the nondimensional density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$ such that $\nabla \cdot \mathbf{u} = 0$, and pressure $p(\mathbf{x}, t)$. With the appropriately defined Reynolds number Re and Prandtl number Pr, and assuming that gravity acts in the direction opposite the unit vector $\mathbf{e}_y$, the nonlinear Boussinesq equation is

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \mathbf{u}\end{bmatrix} = \mathscr{N}\left(\begin{bmatrix}\rho \\ \mathbf{u}\end{bmatrix}\right) := \begin{bmatrix} -\mathbf{u}\cdot\nabla\rho + \dfrac{\nabla^2\rho}{Re\,Pr} \\ -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \dfrac{\nabla^2\mathbf{u}}{Re} - \dfrac{\rho\mathbf{e}_y}{Fr^2}\end{bmatrix}. \tag{1}$$

A steady-state flow is some $\begin{bmatrix}\rho_0(\mathbf{x}) & \mathbf{u}_0(\mathbf{x})^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$ satisfying $\mathscr{N}\left(\begin{bmatrix}\rho_0(\mathbf{x}) & \mathbf{u}_0(\mathbf{x})^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\right) = \mathbf{0}$. The density and velocity fields can be written in terms of perturbations $\rho'(\mathbf{x}, t) := \rho(\mathbf{x}, t) - \rho_0(\mathbf{x})$ and $\mathbf{u}'(\mathbf{x}, t) := \mathbf{u}(\mathbf{x}, t) - \mathbf{u}_0(\mathbf{x})$. If $\|\rho'\| \ll 1$ and $\|\mathbf{u}'\| \ll 1$, then the linearized dynamics are given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho' \\ \mathbf{u}'\end{bmatrix} = \mathscr{L}\begin{bmatrix}\rho' \\ \mathbf{u}'\end{bmatrix} := \begin{bmatrix} -\mathbf{u}_0\cdot\nabla\rho' - \mathbf{u}'\cdot\nabla\rho_0 + \dfrac{\nabla^2\rho'}{Re\,Pr} \\ -\mathbf{u}_0\cdot\nabla\mathbf{u}' - \mathbf{u}'\cdot\nabla\mathbf{u}_0 - \nabla p' + \dfrac{\nabla^2\mathbf{u}'}{Re} - \dfrac{\rho'\mathbf{e}_y}{Fr^2}\end{bmatrix}, \tag{2}$$

subject to $\nabla \cdot \mathbf{u}' = 0$. Next, given the complex conjugation operator $\overline{(\cdot)}$ and some control volume Ω, we define the inner products

$$\langle \rho', \hat{\rho}' \rangle := \int_\Omega \rho' \bar{\hat{\rho}}'\, dV, \quad \langle \mathbf{u}', \hat{\mathbf{u}}' \rangle := \int_\Omega \mathbf{u}' \cdot \bar{\hat{\mathbf{u}}}'\, dV, \quad \left\langle \begin{bmatrix}\rho' \\ \mathbf{u}'\end{bmatrix}, \begin{bmatrix}\hat{\rho}' \\ \hat{\mathbf{u}}'\end{bmatrix} \right\rangle := \langle \rho', \hat{\rho}' \rangle + \langle \mathbf{u}', \hat{\mathbf{u}}' \rangle. \tag{3}$$

By definition, an adjoint of $\mathscr{L}$ with respect to the last inner product is some $\mathscr{L}^*$ with an associated set of conditions on the boundary $\partial\Omega$ of Ω such that, given additional fields $\hat{\rho}'$ and $\hat{\mathbf{u}}'$,

$$\left\langle \mathscr{L}\begin{bmatrix}\rho' \\ \mathbf{u}'\end{bmatrix}, \begin{bmatrix}\hat{\rho}' \\ \hat{\mathbf{u}}'\end{bmatrix} \right\rangle = \left\langle \begin{bmatrix}\rho' \\ \mathbf{u}'\end{bmatrix}, \mathscr{L}^*\begin{bmatrix}\hat{\rho}' \\ \hat{\mathbf{u}}'\end{bmatrix} \right\rangle. \tag{4}$$

It can be shown that this adjoint operator and its associated partial differential equation are

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{\rho}' \\ \hat{\mathbf{u}}'\end{bmatrix} = \mathscr{L}^*\begin{bmatrix}\hat{\rho}' \\ \hat{\mathbf{u}}'\end{bmatrix} = \begin{bmatrix} \mathbf{u}_0\cdot\nabla\hat{\rho}' + \dfrac{\nabla^2\hat{\rho}'}{Re\,Pr} - \dfrac{\hat{\mathbf{u}}'\cdot\mathbf{e}_y}{Fr^2} \\ -\hat{\rho}'\nabla\rho_0 - (\nabla\mathbf{u}_0)\cdot\hat{\mathbf{u}}' + \mathbf{u}_0\cdot\nabla\hat{\mathbf{u}}' - \nabla\hat{p}' + \dfrac{\nabla^2\hat{\mathbf{u}}'}{Re}\end{bmatrix} \tag{5}$$

(see (21) in Ortiz et al., 2015); for clarity, if $\mathbf{u}_0$ and $\hat{\mathbf{u}}'$ are given in indicial notation by u_{0i} and $\hat{u}'_i$, then $((\nabla \mathbf{u}_0)\cdot\hat{\mathbf{u}}')_j = (\partial u_{0i}/\partial x_j)\hat{u}'_i$. The adjoint velocity perturbation is subject to $\nabla\cdot\hat{\mathbf{u}}' = 0$. Given

$$\mathbf{J} := -(\rho'\bar{\hat{\rho}}' + \mathbf{u}'\cdot\bar{\hat{\mathbf{u}}}')\mathbf{u}_0 - p'\bar{\hat{\mathbf{u}}}' + \bar{\hat{p}}'\mathbf{u}' + \frac{1}{Re}\left(\frac{1}{Pr}\left(\bar{\hat{\rho}}'\nabla\rho' - \rho'\nabla\bar{\hat{\rho}}'\right) + (\nabla\mathbf{u}')\cdot\bar{\hat{\mathbf{u}}}' - (\nabla\bar{\hat{\mathbf{u}}}')\cdot\mathbf{u}'\right) \tag{6}$$

and the outward unit normal vector $\mathbf{n}$ on $\partial\Omega$, the boundary condition of $\mathscr{L}^*$ is $\mathbf{J}(\mathbf{x})\cdot\mathbf{n} = 0$ for $\mathbf{x}\in\partial\Omega$. To verify this adjoint operator and boundary condition, it can be checked that

$$\left(\mathscr{L}\begin{bmatrix}\rho'\\ \mathbf{u}'\end{bmatrix}\right)\cdot\overline{\begin{bmatrix}\hat{\rho}'\\ \hat{\mathbf{u}}'\end{bmatrix}} = \begin{bmatrix}\rho'\\ \mathbf{u}'\end{bmatrix}\cdot\overline{\mathscr{L}^*\begin{bmatrix}\hat{\rho}'\\ \hat{\mathbf{u}}'\end{bmatrix}} + \nabla\cdot\mathbf{J}. \tag{7}$$

Integrating (7) over Ω, employing the divergence theorem on the last term, and applying the boundary condition, we recover the definition of the adjoint (4).

To formulate the sensitivity, we first consider the direct and adjoint eigendecompositions

$$\mathscr{L}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}, \quad \mathscr{L}^*\boldsymbol{\psi} = \bar{\lambda}\boldsymbol{\psi}, \qquad \text{where} \qquad \boldsymbol{\phi} = \begin{bmatrix}\phi_\rho\\ \boldsymbol{\phi}_\mathbf{u}\end{bmatrix}, \quad \boldsymbol{\psi} = \begin{bmatrix}\psi_\rho\\ \boldsymbol{\psi}_\mathbf{u}\end{bmatrix} \tag{8}$$

are the direct and adjoint eigenmodes; the ρ and $\mathbf{u}$ subscripts respectively denote the density and velocity components. As usual, $\Re(\lambda)$ and $\Im(\lambda)$ respectively indicate the growth rate and frequency of $\boldsymbol{\phi}$. Given the construction of $\mathscr{L}$ and $\mathscr{L}^*$, we assume that $\nabla\cdot\boldsymbol{\phi} = \nabla\cdot\boldsymbol{\psi} = 0$. The stability sensitivity addresses what eigenvalue perturbation $\delta\lambda$ would result from an infinitesimal change $\delta\mathscr{L}$ in $\mathscr{L}$. An infinitesimal perturbation of the direct eigendecomposition yields $\delta\mathscr{L}\,\boldsymbol{\phi} + \mathscr{L}\,\delta\boldsymbol{\phi} = \delta\lambda\,\boldsymbol{\phi} + \lambda\,\delta\boldsymbol{\phi}$, but $\delta\boldsymbol{\phi}$ is unknown. By taking the inner product of this equation with $\boldsymbol{\psi}$, however, we can take advantage of the fact that $\langle\mathscr{L}\,\delta\boldsymbol{\phi},\boldsymbol{\psi}\rangle = \langle\delta\boldsymbol{\phi},\mathscr{L}^*\boldsymbol{\psi}\rangle = \langle\delta\boldsymbol{\phi},\bar{\lambda}\boldsymbol{\psi}\rangle = \langle\lambda\,\delta\boldsymbol{\phi},\boldsymbol{\psi}\rangle$; hence,

$$\delta\lambda = \frac{\langle\delta\mathscr{L}\,\boldsymbol{\phi},\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle}. \tag{9}$$

We first consider the case where $\delta\mathscr{L}$ corresponds to an infinitesimal change in the base flow density from $\rho_0(\mathbf{x})$ to some $\rho_0(\mathbf{x}) + \delta\rho_0(\mathbf{x})$, while $\mathbf{u}_0(\mathbf{x})$ remains fixed. This operator is given by

$$\delta\mathscr{L}\begin{bmatrix}\rho'\\ \mathbf{u}'\end{bmatrix} = \begin{bmatrix}-\mathbf{u}'\cdot\nabla\delta\rho_0\\ \mathbf{0}\end{bmatrix}, \tag{10}$$

such that the resulting eigenvalue perturbation (9) is

$$\begin{aligned}\delta\lambda &= \frac{\langle -\boldsymbol{\phi}_\mathbf{u}\cdot\nabla\delta\rho_0, \psi_\rho\rangle}{\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle} && \text{(11a)}\\ &= \frac{1}{\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle}\int_\Omega -\bar{\psi}_\rho\boldsymbol{\phi}_\mathbf{u}\cdot\nabla\delta\rho_0\,dV && \text{(11b)}\\ &= \frac{1}{\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle}\left(\int_\Omega \delta\rho_0\boldsymbol{\phi}_\mathbf{u}\cdot\nabla\bar{\psi}_\rho\,dV - \oint_{\partial\Omega}\delta\rho_0\bar{\psi}_\rho\boldsymbol{\phi}_\mathbf{u}\cdot\mathbf{n}\,dS\right), && \text{(11c)}\end{aligned}$$

where we have used integration by parts to obtain (11c). Hence, if

$$\delta\rho_0(\mathbf{x})\bar{\psi}_\rho(\mathbf{x})\boldsymbol{\phi}_\mathbf{u}(\mathbf{x})\cdot\mathbf{n} = 0 \quad \text{for } \mathbf{x}\in\partial\Omega, \tag{12}$$

then we can define the *partial sensitivity to base flow density modifications*

$$\frac{\partial \lambda}{\partial \rho_0} := \frac{\boldsymbol{\phi}_{\mathbf{u}} \cdot \nabla \bar{\psi}_\rho}{\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle}, \quad \text{such that} \quad \delta\lambda = \left\langle \frac{\partial \lambda}{\partial \rho_0}, \delta\rho_0 \right\rangle. \tag{13}$$

The sensitivity is denoted as a *partial* one, because $\mathbf{u}_0$ has remained fixed. The utility of (13) is that the change in the stability eigenvalue can be efficiently *a priori* computed for any infinitesimal base flow density shift $\delta\rho_0$ satisfying the boundary condition (12).

Next, the same derivation can be done for the opposite case, where $\rho_0(\mathbf{x})$ remains fixed but the base flow velocity is infinitesimally shifted from $\mathbf{u}_0(\mathbf{x})$ to $\mathbf{u}_0(\mathbf{x}) + \delta\mathbf{u}_0(\mathbf{x})$. The shift in $\mathscr{L}$ is

$$\delta\mathscr{L} \begin{bmatrix} \rho' \\ \mathbf{u}' \end{bmatrix} = \begin{bmatrix} -\delta\mathbf{u}_0 \cdot \nabla\rho' \\ -\delta\mathbf{u}_0 \cdot \nabla\mathbf{u}' - \mathbf{u}' \cdot \nabla\delta\mathbf{u}_0 \end{bmatrix}, \tag{14}$$

and the shift in the eigenvalue is

$$\delta\lambda = \frac{\langle -\delta\mathbf{u}_0 \cdot \nabla\phi_\rho, \psi_\rho \rangle + \langle -\delta\mathbf{u}_0 \cdot \nabla\boldsymbol{\phi}_{\mathbf{u}} - \boldsymbol{\phi}_{\mathbf{u}} \cdot \nabla\delta\mathbf{u}_0, \boldsymbol{\psi}_{\mathbf{u}} \rangle}{\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle} \tag{15a}$$

$$= \frac{1}{\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle} \int_\Omega \left(-\bar{\psi}_\rho \delta\mathbf{u}_0 \cdot \nabla\phi_\rho - (\delta\mathbf{u}_0 \cdot \nabla\boldsymbol{\phi}_{\mathbf{u}}) \cdot \bar{\boldsymbol{\psi}}_{\mathbf{u}} - (\boldsymbol{\phi}_{\mathbf{u}} \cdot \nabla\delta\mathbf{u}_0) \cdot \bar{\boldsymbol{\psi}}_{\mathbf{u}} \right) dV \tag{15b}$$

$$= \frac{1}{\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle} \left(\int_\Omega \left(-\bar{\psi}_\rho \delta\mathbf{u}_0 \cdot \nabla\phi_\rho - (\delta\mathbf{u}_0 \cdot \nabla\boldsymbol{\phi}_{\mathbf{u}}) \cdot \bar{\boldsymbol{\psi}}_{\mathbf{u}} + (\boldsymbol{\phi}_{\mathbf{u}} \cdot \nabla\bar{\boldsymbol{\psi}}_{\mathbf{u}}) \cdot \delta\mathbf{u}_0 \right) dV - \oint_{\partial\Omega} (\delta\mathbf{u}_0 \cdot \bar{\boldsymbol{\psi}}_{\mathbf{u}})(\boldsymbol{\phi}_{\mathbf{u}} \cdot \mathbf{n})\, dS \right). \tag{15c}$$

Hence, if

$$(\delta\mathbf{u}_0(\mathbf{x}) \cdot \bar{\boldsymbol{\psi}}_{\mathbf{u}}(\mathbf{x}))(\boldsymbol{\phi}_{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{n}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega, \tag{16}$$

then we can define the *partial sensitivity to base flow velocity modifications*

$$\frac{\partial \lambda}{\partial \mathbf{u}_0} := \frac{-\bar{\psi}_\rho \nabla\phi_\rho - (\nabla\boldsymbol{\phi}_{\mathbf{u}}) \cdot \bar{\boldsymbol{\psi}}_{\mathbf{u}} + \boldsymbol{\phi}_{\mathbf{u}} \cdot \nabla\bar{\boldsymbol{\psi}}_{\mathbf{u}}}{\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle}, \quad \text{such that} \quad \delta\lambda = \left\langle \frac{\partial \lambda}{\partial \mathbf{u}_0}, \delta\mathbf{u}_0 \right\rangle. \tag{17}$$

Again, (17) provides an efficient way to compute the change in λ for any $\delta\mathbf{u}_0$ satisfying (16).

3 Methods

The three Boussinesq operators (1, 2, 5) are computed using a finite-volume formulation based on OpenFOAM (Weller et al., 1998). To compute the steady state flows, a time-resolved nonlinear solver is run until the root-mean-square of the output of $\mathscr{N}$ is $8 \cdot 10^{-5}$ or smaller. The leading eigenmodes are then computed using the Arnoldi iteration with 3,600 to 5,400 iterations.

In our example, a two-dimensional horizontal flow encounters a thin plate occupying $[-0.025, 0.025] \times [-0.5, 0.5]$, where the entire numerical domain is $[-100, 100] \times [-100, 100]$. At the left boundary, we set $\mathbf{u} = \mathbf{e}_x$ (i.e., the unit horizontal vector) and we fix ρ to be linearly stratified. On the plate, $\mathbf{u} = \mathbf{0}$ and $\mathbf{n} \cdot \nabla\rho^* = 0$, where ρ^* is the density perturbation from the background stratification. Outflow conditions are used elsewhere. Neumann pressure conditions are set on all boundaries to be consistent with the momentum equation, except at the outlet, where $p = 0$. The boundary conditions of the linearized operator (2) are homogeneous variants of the above, and the boundary conditions of the adjoint operator (5) are described in Section 2.

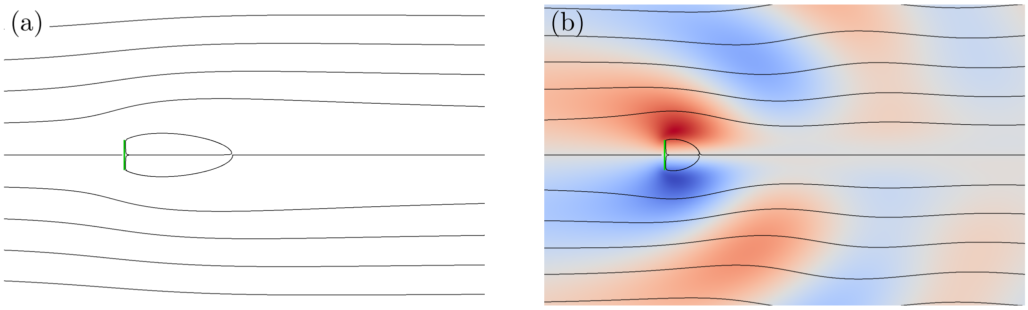

Figure 1: Two-dimensional flows from left to right around a vertical thin plate (green), at $Re = 30$. (a) The unstratified flow ($Fr = \infty$), shown as streamlines; the density field is uniform. (b) $Fr = 1$ and $Pr = 7.19$, with color indicating the density perturbation from the background linear stratification (red: positive; blue: negative).

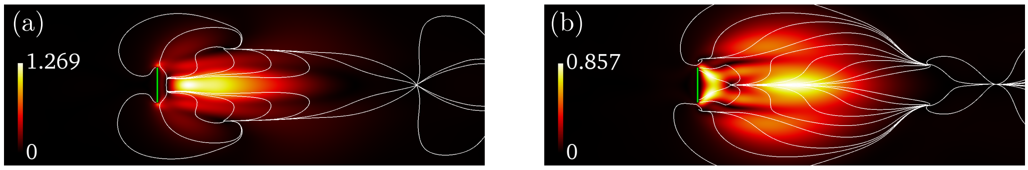

Figure 2: The partial sensitivities of the least stable eigenvalue $\lambda = -0.012 + 0.642i$ of the $Fr = \infty$ flow, shown as pointwise vector magnitudes of (a) $\Re(\partial\lambda/\partial\mathbf{u}_0)$ (streamlines enter the recirculation region) and (b) $\Im(\partial\lambda/\partial\mathbf{u}_0)$ (streamlines exit the recirculation region).

4 Example: two-dimensional flow around a finite transverse thin plate

The stable base flow at $Re = 30$ is shown in Figure 1 for $Fr = \infty$ (i.e., unstratified) and $Fr = 1$ (i.e., strongly stratified). The Prandtl number in the stratified simulation is 7.19, which is roughly representative of oceanographic flows. The unstratified base flow exhibits a long recirculation bubble, whereas the stratified flow has a shorter recirculation bubble because of the vertically restoring buoyancy forces. Furthermore, lee waves with a wavelength of $2\pi Fr$ are present, as predicted by inviscid theory (Turner, 1973).

In the unstratified flow, the leading eigenvalue is $\lambda = -0.012 \pm 0.642i$. For this eigenvalue, $\psi_\rho = 0$; therefore, the stability can be analyzed the same way as the constant-density flow. That is, $\partial\lambda/\partial\rho_0 = 0$, and the first summand of (17) is also trivial. The sensitivity analysis proceeds exactly as described by Marquet et al. (2008), and is shown as growth rate and frequency sensitivities in Figure 2. It is immediately apparent that both sensitivities are largest in and around the recirculation bubble, and nearly zero everywhere else. Also, the maximally destabilizing base flow modification is the addition of upstream velocity in the recirculation region, which would strengthen the vorticity created at the plate tips. These results are in excellent agreement with the sensitivity of the cylinder flow computed by Marquet et al.

In contrast, for the $Fr = 1$ flow, the partial sensitivities of the least stable eigenvalue $\lambda = -0.001 - 0.902i$ to base flow density and velocity modifications are shown in Figure 3. The partial sensitivity to base flow density modifications (Figure 3(a, b)) is largest upstream of the bluff body. Such a feature is in stark contrast to constant-density sensitivity analyses, which place sensitivity regions downstream as in Figure 2. The upstream placement of $\partial\lambda/\partial\rho_0$, however, may be intuitive given that stratified flows can exhibit upstream wakes in the form of blocking (Turner, 1973).

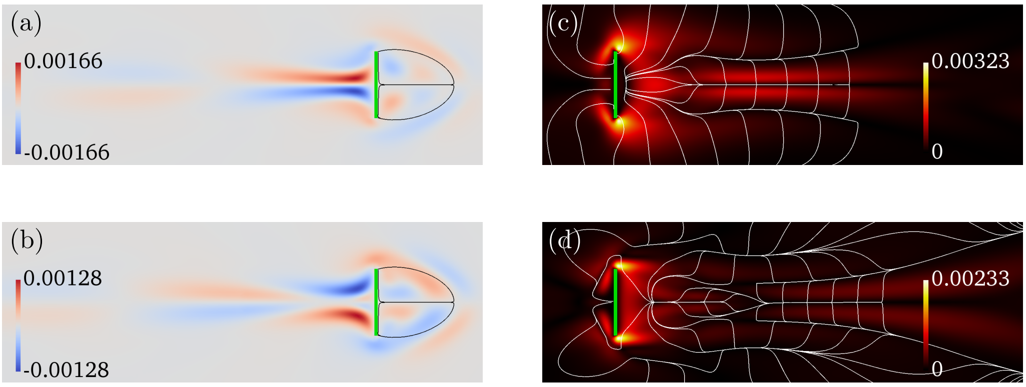

Figure 3: The partial sensitivities of the $Fr = 1$ flow, for the least stable eigenvalue $\lambda = -0.001 - 0.902i$. The sensitivities are shown as (a) $\Re(\partial\lambda/\partial\rho_0)$ and (b) $\Im(\partial\lambda/\partial\rho_0)$. Also shown are pointwise vector magnitudes of (c) $\Re(\partial\lambda/\partial\mathbf{u}_0)$, and (d) $\Im(\partial\lambda/\partial\mathbf{u}_0)$. Streamlines enter the recirculation region in (c, d). The recirculation region is shown in (a, b).

The partial sensitivity to base flow velocity modifications (Figure 3(c, d)) is primarily a downstream structure, as in the unstratified flow. Significant differences are evident, however. First, $\partial\lambda/\partial\mathbf{u}_0$ has a nontrivial upstream presence because of the component from $-\bar{\psi}_\rho \nabla\phi_\rho$ (not shown; see (17)), which is ultimately derived from $-\delta\mathbf{u}_0 \cdot \nabla\phi_\rho$ (i.e., the transport of density perturbations by $\delta\mathbf{u}_0$; see (14)). Second, $\partial\lambda/\partial\mathbf{u}_0$ is large not just everywhere in the recirculation bubble, but specifically near the tips of the plate, where vertical buoyancy forces are particularly large. The streamlines of $\Re(\partial\lambda/\partial\mathbf{u}_0)$ indicate that an addition of velocity compressing the recirculation bubble would be maximally destabilizing.

5 Conclusion

A novel theory describing the sensitivities of linear Boussinesq stability to base flow density and velocity modifications has been presented, by extending the constant-density theory of Marquet et al. (2008). Though it is broadly applicable to Boussinesq flows, its application to density-stratified flows is particularly revealing, as it demonstrates sensitivity features not seen in constant-density flows. Whereas constant-density theory shows that base flow velocity modifications in recirculation regions are maximally stabilizing or destabilizing, the stratified analysis shows that base flow density modifications upstream of bluff bodies are maximally stabilizing or destabilizing. Furthermore, the effects of base flow velocity modifications are altered in the presence of density stratification.

We are currently preparing a journal article which presents the full details of our analytical development and numerical examples. The article also examines the global direct and adjoint eigenmodes of the examples, explores the weakly stratified flow at $Fr = 8$, and analyzes the sensitivities in much greater depth. The ultimate aim of the sensitivity analysis is to provide a new avenue through which the stability of oceanographic flow structures may be understood.

Acknowledgements

K. K. C. was supported by the Viterbi Postdoctoral Fellowship, provided by the Viterbi

School of Engineering at the University of Southern California (USC). G. R. S. gratefully acknowledges the support from ONR N00014-15-1-2506, administered by Dr. R. Joslin. The computations were performed using resources provided by USC's Center for High-Performance Computing.

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