Reduced Modeling of Strongly Stratified Turbulence

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Strongly stratified turbulent shear flows are of fundamental importance owing to their widespread occurrence and their impact on diabatic mixing. Stable stratification in high Reynolds number (Re) flows drives anisotropization of motions having horizontal scales L larger than the Ozmidov scale $l_O \equiv \sqrt{\varepsilon/N^3}$ (where ε is the energy dissipation rate and N is the Brunt frequency), below which buoyancy forces are negligible. The mechanisms by which energy is transferred from these large-scale quasi-horizontal (quasi-2D) 'pancake' modes to smaller scales and the associated interplay between anisotropic and isotropic dynamics remain ill understood. DNS is particularly challenging owing to the extreme range of spatiotemporal scales that must be resolved. Indeed, in terms of the Froude number $Fr \equiv U/(NL)$ ($\simeq 10^{-3}$ in geophysical flows), where U is a characteristic large-scale horizontal velocity, the ratio $l_O/L = O(Fr^{3/2}) \ll 1$ [1,2].

Here, we exploit the limit $Fr \to 0$ and the corresponding strong anisotropy exhibited by the large-scale motions to derive a reduced PDE model of stratified turbulence. We employ multiple-scale asymptotics [3] by considering a distinguished limit in which the buoyancy Reynolds number $\mathcal{R} \equiv Fr^2 Re$, a measure of the level of turbulence within the pancakes, is fixed and the aspect-ratio of the large-scale flow is $O(Fr)$ as $Fr \to 0$, implying the relevant vertical length scale $h = O(U/N)$ [4]. All flow variables are allowed to depend on fast and slow horizontal and temporal coordinates ($\boldsymbol{\chi}$,τ) and ($\mathbf{x}$,t), respectively, as well as on a single vertical coordinate z. Each generic field ϕ is then decomposed into a fast ($\boldsymbol{\chi}$,τ) mean and a fluctuation:

$$\phi(\mathbf{x}, z, t) \to \phi(\boldsymbol{\chi}, \mathbf{x}, z, \tau, t) = \overline{\phi}(\mathbf{x}, z, t) + \phi'(\boldsymbol{\chi}, \mathbf{x}, z, \tau, t), \text{ where } \overline{\phi'} \equiv 0.$$

Introducing $\epsilon \equiv \sqrt{Fr}$, the following asymptotic expansions are posited for the various fields:

$$[\mathbf{u}, b, p] \quad \sim \quad [\mathbf{u}_0, b_0, p_0] + \epsilon[\mathbf{u}_1, b_1, p_1] + \epsilon^2[\mathbf{u}_2, b_2, p_2] + \ldots, \quad W \sim \epsilon^{-1} W_{-1} + W_0 + \epsilon W_1 + \ldots$$

The crucial prescription is that the vertical velocity W (normalized by hU/L) is no larger than $O(\epsilon^{-1})$ on fine horizontal scales. This re-scaling ensures a consistent dominant balance among the feedback of the fluctuations on the mean fields through the vertical Reynolds stress divergence $\partial_z(\overline{W'\mathbf{u}'})$; the mean tendency $\partial_t \overline{\mathbf{u}}$; and mean vertical diffusion $\mathcal{R}^{-1}\partial_z^2\overline{\mathbf{u}}$. It can then be deduced that the fluctuating horizontal velocity, buoyancy, and pressure fields arise at $O(\epsilon)$, a key simplification. Substituting these expansions into the Boussinesq equations, collecting terms order by order in ϵ, and eliminating secular growth terms yields a closed, coupled system of PDEs for the leading-order (omitting subscripts) mean and fluctuation fields:

$$\begin{aligned}
\left(\partial_t + \overline{\mathbf{u}} \cdot \nabla_{\mathbf{x}} + \overline{W}\partial_z\right)\overline{\mathbf{u}} + \partial_z\left(\overline{W'\mathbf{u}'}\right) &= -\nabla_{\mathbf{x}}\overline{p} + \frac{1}{\mathcal{R}}\partial_z^2\overline{\mathbf{u}}, &(1)\\
0 &= -\partial_z\overline{p} + \overline{b}, &(2)\\
\left(\partial_t + \overline{\mathbf{u}} \cdot \nabla_{\mathbf{x}} + \overline{W}\partial_z\right)\overline{b} + \partial_z\left(\overline{W'b'}\right) &= \frac{1}{Pr\mathcal{R}}\partial_z^2\overline{b}, &(3)\\
\nabla_{\mathbf{x}} \cdot \overline{\mathbf{u}} + \partial_z\overline{W} &= 0; &(4)\\
\left(\partial_\tau + \overline{\mathbf{u}} \cdot \nabla_{\boldsymbol{\chi}}\right)\mathbf{u}' + W'\partial_z\overline{\mathbf{u}} &= -\nabla_{\boldsymbol{\chi}}p' + \frac{Fr}{\mathcal{R}}\left(\nabla_{\boldsymbol{\chi}}^2 + \partial_z^2\right)\mathbf{u}', &(5)\\
\left(\partial_\tau + \overline{\mathbf{u}} \cdot \nabla_{\boldsymbol{\chi}}\right)W' &= -\partial_z p' + b' + \frac{Fr}{\mathcal{R}}\left(\nabla_{\boldsymbol{\chi}}^2 + \partial_z^2\right)W', &(6)\\
\left(\partial_\tau + \overline{\mathbf{u}} \cdot \nabla_{\boldsymbol{\chi}}\right)b' + W'\partial_z\overline{\mathbf{b}} &= \frac{Fr}{Pr\mathcal{R}}\left(\nabla_{\boldsymbol{\chi}}^2 + \partial_z^2\right)b', &(7)\\
\nabla_{\boldsymbol{\chi}} \cdot \mathbf{u}' + \partial_z W' &= 0. &(8)
\end{aligned}$$

The mean system (1)–(4) is recognizable as the 'hydrostatic primitive equations' augmented with the vertical divergence of Reynolds stresses and buoyancy fluxes arising from the *isotropic, non-hydrostatic* fluctuation dynamics. Crucially, the usual closure difficulties associated with Reynolds averaging are circumvented here by exploiting the scale separation that emerges as $Fr \to 0$ and by setting the dimensional

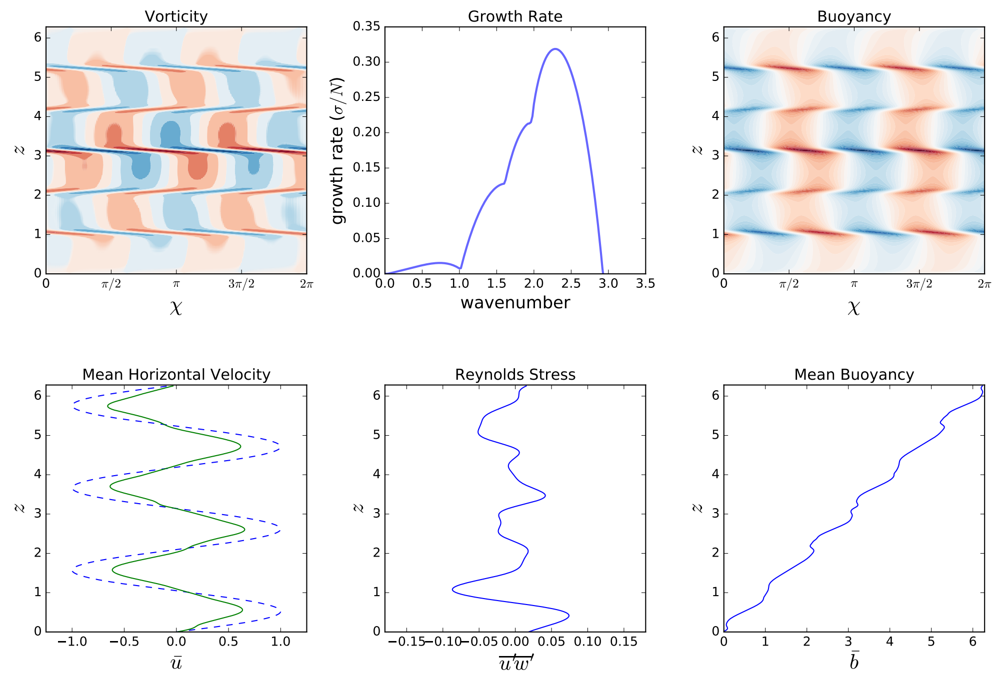

Figure 1: 2D response to sinusoidal forcing $\bar{f}_x = (m^2/\mathcal{R})\sin(mz)$, where $m = 3$, $\mathcal{R} = 200$, $Fr = 0.02$. Upper: Linear stability results. Lower: Snapshot of a time-integration of the reduced system.

amplitude of the fluctuating velocity components to be $O(\sqrt{Fr}U)$. This scaling is compatible with the assertion in Riley & Lindborg [2] that the leading spectral contribution to the *mean square* vertical velocity in stratified turbulence arises at small scales and has a magnitude $O(FrU^2)$. To illustrate the properties of the reduced equations, a 2D flow driven by a vertically-varying sinusoidal body force $[(m^2/\mathcal{R})\sin(mz)]$ in the presence of a linear ambient buoyancy profile $\bar{b}_L(z) = z$ is considered. In the absence of instabilities, this force would drive a laminar shear flow $\overline{u}_L(z) = \sin(mz)$, but as shown in Fig. 1 this laminar mean state is linearly unstable to χ-varying perturbations for a range of physically-relevant values of the parameters (m, $\mathcal{R}$, Fr). Consequently, an initial-value computation of the reduced equations shows sustained non-trivial nonlinear dynamics, in which the fluctuation fields drive the mean horizontal flow away from $\overline{u}_L(z)$ and the mean buoyancy profile develops a staircase-like structure with internal mixed layers (see Fig. 1). These emergent well-mixed regions can be shown to have a thickness on the order of the Ozmidov scale.

It is instructive to note that the fluctuation system is *quasi-linear* (QL) about the local mean fields, suggesting that the 2nd-order cumulant expansion approach used, e.g., by Tobias & Marston [5] for reduced modeling of other anisotropic geophysical flows may be asymptotically justified for strongly stratified turbulence. By retaining multiple horizontal and temporal scales, the multiscale reduced system (1)–(8) in fact *extends* the traditional QL reduction by allowing for slow spatiotemporal evolution of the large-scale fields in accord with their own fully nonlinear dynamics. This generalization of the QL approximation has inspired the so-called GQL formalism, which recently has been shown to be significantly more accurate than QL schemes [6]. To capture certain nonlinear fluctuation/fluctuation interactions, which can attain leading-order significance within regions in which vertical spatial gradients are amplified (e.g. critical layers, see Fig. 1), the multiple-scales analysis can be complemented with a matched asymptotic analysis, yielding a novel multiscale formulation that goes beyond the traditional or even generalized QL approximation.

References

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