Internal-Gravity wave propagation in a range-dependent waveguide with forcing at the bottom

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Abstract

We compute the propagation of waves forced at the bottom of an inhomogeneous stratified moving fluid, using vertical normal modes and horizontal rays. The problem is separated into a range-dependent nonlinear eigenproblem that defines the vertical structure of the solution and a set of coupled inhomogeneous ordinary differential equations with non-constant coefficients in the horizontal. Under the adiabatic approximation, i.e. neglecting the energy transfer between different modes due to variation of the medium properties in the horizontal, the latter reduces to a set of uncoupled ODEs. Results of the implementation of the method are given in an idealized case.

1 Introduction

A common property of ocean, atmospheric and other geophysical systems, is that both their vertical extent and the typical vertical lengthscale of variation of the medium's properties are small compared to the horizontal ones. If the fluid is vertically bounded (e.g. by physical boundaries) or if there is a mechanism that results in a vertical trapping of the propagating waves, the physical system forms a waveguide, in which constructive and destructive interferences of vertically bouncing waves result in a discrete set of vertical modes that propagate horizontally (i.e. along the waveguide). This is the approach we take here, which couples range-dependent normal modes and horizontal rays.

While this method has been widely used in ocean acoustics (e.g. Burridge and Weinberg, 1977; Desaubies and Dysthe, 1995; Jensen et al., 2011) and in atmospheric infrasound propagation (Brekhovskikh and Godin, 1999; LePichon et al., 2009), it has received relatively little attention in the context of internal wave propagation, especially with a moving background state. Notable exceptions are the work of Keller and Mow (1969), who used a very similar approach to study the internal-wave dynamics in an ocean with a sloping bottom, and more recent papers by Godin (2002, 2012), who gave a generalized formulation for acoustic-gravity waves in range-dependent waveguide with a background velocity field. Most studies (if not all) address the free propagation of waves (with eventually attenuation) or the response to a point source in the waveguide. In many geophysical situations, such as mountain-generated waves in the atmosphere or tidal conversion by a ridge in the ocean, waves are forced at the boundary (usually the bottom) over a finite region in space.

In this paper, we derive a similar set of equations in a moving fluid under the Boussinesq approximation, describing the steady-state response to a forcing mechanism at the bottom of the domain. The latter is explicitly included in the formulation and has a finite size in the horizontal. After deriving the linearized equation and expressing the solution in terms of range-dependent vertical modes and a set of coupled-mode equations (§ 2), we give some results of the numerical implementation of the method, in a case where a stratified fluid with a surface-intensified jet is forced by the oscillation of a bottom disturbance (§ 3). A brief discussion is given in § 4.

2 Formulation of the coupled mode equations

We first derive the linear theory for a range-dependent waveguide using the Boussinesq approximation. Detailed calculations are omitted here for the sake of brevity, and only the major steps of the procedure are given.

2.1 Linearized Boussinesq equation in Lagrangian variables

We start with the standard Boussinesq equations linearized around a background state with velocity $\vec{V}(x,y,z) = (U\vec{e_x}, V\vec{e_y}, W\vec{e_z})$, pressure (geopotential) $P_0(x,y,z)$ and buoyancy $B(x,y,z) = -g\rho_0(x,y,z)/\hat{\rho}$ where $\hat{\rho}$ is the reference density:

$$(\partial_t + \vec{V}\cdot\vec{\nabla})\tilde{\vec{v}} + (\tilde{\vec{v}}\cdot\vec{\nabla})\vec{V} + \vec{\nabla}p - b\vec{e_z} = 0, \tag{1}$$

$$(\partial_t + \vec{V}\cdot\vec{\nabla})b + W\partial_z b + \vec{N^2}\cdot\tilde{\vec{v}} = 0, \tag{2}$$

$$\vec{\nabla}\cdot\tilde{\vec{v}} = 0. \tag{3}$$

Here, tildes denotes Eulerian velocities, and we have defined a Brunt-Väisälä frequency vector $\vec{N^2} \equiv (N_x^2\vec{e_x}, N_y^2\vec{e_y}, N^2\vec{e_z}) = \vec{\nabla}B$. Note that N_x^2 and N_y^2 are not necessarily positive (but we want to keep the notation consistent with the standard Brunt-Väisälä frequency). The steady background state is assumed to be hydrostatic and obeys the following equations:

$$\partial_z P_0 = B, \qquad \vec{\nabla}\cdot\vec{V} = 0, \qquad \vec{V}\cdot\vec{\nabla}B = 0. \tag{4}$$

Note that because of the hydrostatic approximation, the basic state is not an exact solution of the initial system of equations. However, the associated error is small. A non-hydrostatic background state could be used, but this would make the following derivation artificially more involved while not changing the results fundamentally.

We recast the linearized system of equations in terms of particle displacement $\vec{\delta x}$ and particle velocity $\vec{v}$ (Godin, 1997). These are related by the following linearized relation:

$$\vec{v} \equiv \frac{\mathrm{d}\vec{\delta x}}{\mathrm{d}t} = (\partial_t + \vec{V}\cdot\vec{\nabla})\vec{\delta x} = \tilde{\vec{v}} + (\vec{\delta x}\cdot\vec{\nabla})\vec{V}, \tag{5}$$

where $\mathrm{d}(\cdot)/\,\mathrm{d}t \equiv \partial_t + (\vec{V}\cdot\vec{\nabla})$ is the (linearized) convective derivative. By definition, the buoyancy reads: $b = -\vec{N^2}\cdot\vec{\delta x}$. The linearized equations thus become:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + (\vec{\delta x}\cdot\vec{\nabla})\vec{\nabla}P_0 + \vec{\nabla}p = 0, \tag{6}$$

$$\vec{\nabla}\cdot\vec{\delta x} = 0. \tag{7}$$

From now on, we will consider a two-dimensional waveguide and retain only the x- and z- spatial coordinates, and will work with the time Fourier-transformed equation.

2.2 Vertical normal modes

From the range-independent version of the linearized equations (6–7), one can expand the variable in Fourier modes in the horizontal and time coordinates and derive a vertical eigenproblem. We choose to formulate this eigenproblem in terms of the functions ϕ and φ corresponding to normal modes of the variables p and δz respectively mode. We thus have:

$$\partial_z \phi = -\hat{\alpha}\varphi, \qquad \partial_z \varphi = \frac{k^2}{\hat{\omega}^2}\phi, \tag{8}$$

where $\hat{\omega} = \omega - Uk$ and $\hat{\alpha} = N^2 - \hat{\omega}^2$. This is a fourth-order polynomial eigenproblem with eigenvalues k_n, which come by pairs $k_n^{\pm}$ associated with slightly different vertical modes with the same number of zeros. For a vanishing background flow, the normal modes are exactly the same within each pair and we have $k_n^+ = -k_n^-$. In the following, we skip the indices $\pm$ except when it is explicitly required. Taking the integral of $\partial_z(\phi_m\varphi_n - \phi_n\varphi_m)$ and after some manipulations, one obtains the following generalized orthonormality condition (for homogeneous Dirichlet or Neumann vertical boundary conditions):

$$\int_z \left[\frac{\omega(\hat{\omega}_n k_m + \hat{\omega}_m k_n)}{\hat{\omega}_n^2\hat{\omega}_m^2}\phi_n\phi_m + U(\hat{\omega}_n + \hat{\omega}_m)\varphi_n\varphi_m\right] \mathrm{d}z = c_n\delta_{n,m}, \tag{9}$$

where c_n is a normalization constant, $\alpha = N^2 - \omega^2$ and $\delta_{n,m}$ the standard Kronecker symbol. We then define the state vector for the variables $(p, \delta x, u, \delta z, w)$ in a normal mode:

$$\Phi_n = \left(\phi_n, \frac{ik_n\phi_n}{\hat{\omega}_n^2}, \frac{k_n\phi_n}{\hat{\omega}_n}, \varphi_n, -i\hat{\omega}_n\varphi_n\right)^{\intercal}.$$

This allows us to recast the previous generalized orthonormality condition in a simple matrix form:

$$\int_z \Psi_{-m}^{\intercal}\mathbb{B}\Phi_n\,\mathrm{d}z = \frac{c_n}{i}\delta_{n,m}, \tag{10}$$

where Ψ_{-m} is the auxiliary state vector (Φ_m with $k \to -k$ and $U \to -U$), and

$$\mathbb{B} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & U & 0 & 0 \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U \\ 0 & 0 & 0 & U & 0 \end{pmatrix}. \tag{11}$$

2.3 Projection of linearized equation onto the vertical modes

Using the same state vector as for the normal modes, $\Phi = (p, \delta x, u, \delta z, w)$, and after some manipulation, the system of equations (6–7) can be written as a first-order system in terms of x-derivatives:

$$\frac{\partial\Phi}{\partial x} = \mathbb{A}\Phi. \tag{12}$$

The matrix $\mathbb{A}$ is not given here to keep the formulation as short as possible, and because it does not enter the derivation explicitly. We next expand the state vector in a series of normal modes: $\Phi = \sum_m \Phi_m(z; x, \omega)F_m(x; \omega)$, insert this form into equation (12), and multiply by $\Psi_{-n}\mathbb{B}$ on the left side, to get:

$$\sum_m \Psi_n^{\intercal}\mathbb{B}\Phi_m\frac{\mathrm{d}F_m}{\mathrm{d}x} = \sum_m \Psi_n^{\intercal}\mathbb{C}\Phi_m F_m - \sum_m \Psi_n^{\intercal}\mathbb{B}\frac{\partial\Phi_m}{\partial x}F_m \tag{13}$$

with

$$\mathbb{C} = \mathbb{B}\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & \partial_z & 0 \\ 0 & -\partial^2_{xx} P_0 & i\omega - W\partial_z & -\partial_x B & 0 \\ 0 & 0 & 0 & -U\partial_z & 0 \\ -\partial_z & -\partial_x B & 0 & -N^2 & i\omega - W\partial_z \\ 0 & 0 & 0 & i\omega - W\partial_z & +1 \end{pmatrix}. \tag{14}$$

The final step consists in the vertical integration of the above equation. When doing so, it is convenient to split the matrix $\mathbb{C}$ into a matrix $\mathbb{C}'$ containing only the terms that do not vanish if the basic state is range-independent and $\mathbb{C}''$ that contains the other terms (with occurrences of W or x-derivative of the basic-state fields). In order to take into account the forcing at the bottom, it is important to notice that the vertical modes do not obey the same boundary conditions as the actual solution of the problem. In particular, at the bottom, we have $\delta z(x, 0) = B(x)$ where $B(x)$ is the topography, whereas $\varphi(0) = 0$ for a rigid bottom. Therefore, one must beware of swapping vertical derivation and summation over the modes. In the product $\Psi_{-n}\mathbb{C}'\Phi$, each term implying vertical derivative is integrated by parts back and forth, with the normal-mode expansion being made after the first integration. The procedure is as follows:

$$\begin{aligned} \int \frac{\omega}{\hat{\omega}_n}\phi_n \delta z \,\mathrm{d}z + \int \varphi_n \partial_z p \,\mathrm{d}z &= \left[\omega \frac{\phi_n \delta z}{\hat{\omega}_n} + \varphi_n p\right]_{\text{bot.}}^{\text{top}} - \sum_m \left(\omega \int \left(\frac{\phi_n}{\hat{\omega}_n}\right)' \delta z \,\mathrm{d}z - \int \varphi_n' \delta z \,\mathrm{d}z\right) \\ &= \left[\omega \frac{\phi_n \delta z}{\hat{\omega}_n} + \varphi_n p\right]_{\text{bot.}}^{\text{top}} + \sum_m \left(\omega \int \frac{\phi_n \varphi_m'}{\hat{\omega}_n} \,\mathrm{d}z + \int \varphi_n \phi_m' \,\mathrm{d}z\right). \end{aligned} \tag{15}$$

The square bracket term contains the contribution from the forcing at the boundaries. Only the contribution from the bottom is non-zero and will be noted $\beta_n(x)$ in the following.

With further manipulations (Godin, 2002) we obtain from eq. (13) a so-called coupled-mode equation for the modal amplitude F_n:

$$\frac{\mathrm{d}F_n(x)}{\mathrm{d}x} - \left(ik_n - \frac{1}{2k_n}\frac{\mathrm{d}k_n}{\mathrm{d}x}\right) F_n(x) = \frac{\beta_n(x)}{ic_n} + \sum_m g_{nm}(x) F_m(x). \tag{16}$$

The last term contains the coupling terms between modes. They are not explicitly written here for brevity and will be neglected in the following. On the other hand, their investigation may be useful if one is interested in scattering problem, as they drive the exchange of energy between modes. The second term of the LHS results from the integral of $\Psi_{-n}\mathbb{C}'\Phi$. Each pair of eigenvalue $k_n^\pm$ correspond to modal propagation toward the right or the left, depending on whether k_n has the same sign as ω or is oppositely signed, respectively. Hereafter, our convention is that the positive indices indicate modes propagating to the right and conversely. Hence, the boundary condition for equation (16) is $F_n^\pm \to 0$ for $x \to \mp\infty$. We now assume that the horizontal dependence of the medium properties are weak enough for the coupling terms can be neglected. Under this so-called adiabatic approximation, the solution to the amplitude equation for an infinite domain is

$$F_n^\pm(x) = \int_{\mp\infty}^{x} \sqrt{\frac{k_n^\pm(x')}{k_n^\pm(x)}} \frac{\beta_n^\pm(x')}{ic_n^\pm} \exp\left(i\int_{x'}^{x} k_n^\pm(y)\,\mathrm{d}y\right) \mathrm{d}x'. \tag{17}$$

Comparison of this solution with the standard form of WKBJ solution to inhomogeneous 1D wave equation makes clear that the second terms of the LHS of eq. (16) (containing an x-derivative of $k_n(x)$) enforces modal energy conservation.

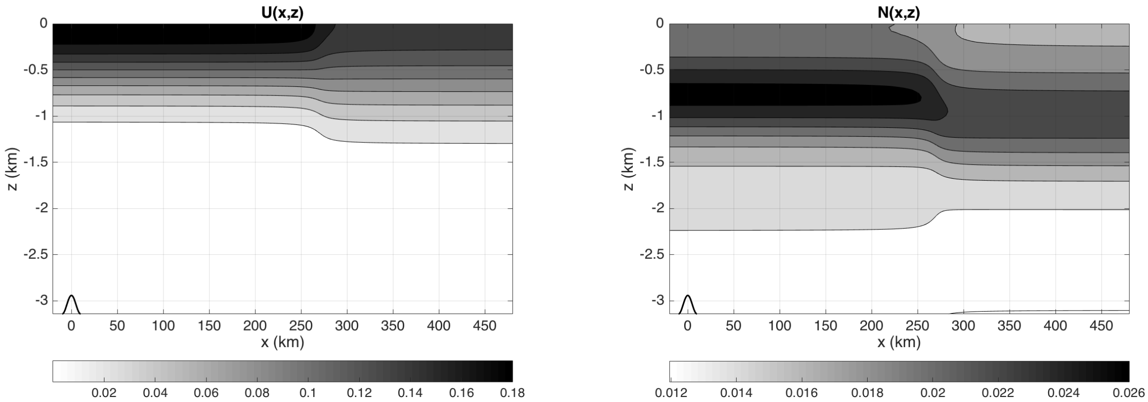

Figure 1: Background state: horizontal velocity (left panel – m/s) and stratification profile (right panel – s^{-1}). The topography generating the waves is plotted in black with arbitrary amplitude.

3 Application: wave beam radiated by an oscillating bottom disturbance

We next present a test case to illustrate the implementation of the method for waves generated by an oscillating topography at the bottom of the domain. This could models oceanic tides over a ridge if one neglects the tidal excursion (Llewellyn Smith and Young, 2002). This is not our goal, however, as the Coriolis effect is not included here. The background jet has a half-Gaussian vertical structure and maximum amplitude at the surface. The stratification profile is a nearly constant profile superimposed with a Gaussian bump with maximum value under the wind jet. These two fields for the case reported here are given in Figure 1. They are computed by first choosing a streamfunction that gives the desired profile for the horizontal velocity. Then, we choose a reference profile for the stratification at $x = 0$ and compute the buoyancy field according to the basic-state equation (rightmost eq. 4). The velocity fields is modulated in range with an arctangent function, centered at $x = 270$ km, implying a weak vertical velocity and a modulation of the stratification profile, according to the basic-state equations (4). The topography is a 20 km width cosine bump centered at $x = 0$ and oscillates at a frequency $\omega = 3 \cdot 10^{-4}\ s^{-1}$.

For the numerical resolution, we use 80 grid points in the vertical and 20 points in the horizontal. Only the lowest 20 vertical modes are retained. The vertical problem defined in (8) is solved at each x-location. The convolution product for the horizontal dependence of the amplitude, given in eq. (17) is solved analytically assuming that the medium properties are range-independent in the region of the forcing. We choose a basic state that ensure this, but in a more general setting, the corresponding integral could be done numerically. In Figure 2, left panel, we show the first and sixth eigenmode ϕ at three different locations. One sees a weak modulation of the mode structure in range, as well as a modulation of the associated eigenvalue. The solution is then computed by interpolating the vertical modes and the eigenvalues on a finer grid, and by adding the different mode contributions. The corresponding results in the case investigated here is given in Figure 2, right panel, for the vertical parcel displacement $\delta z = \sum_n \phi_n(z; x) F_n(x)$.

4 Discussion

We have computed the wave propagation in a range-dependent waveguide, explicitly taking into account the forcing mechanism at the bottom of the waveguide. The formulation

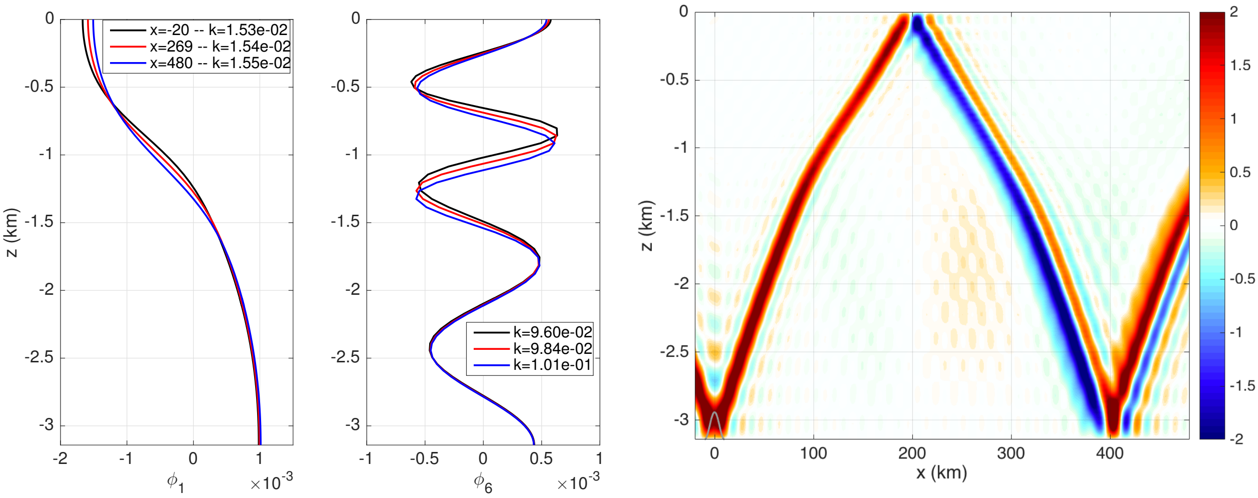

Figure 2: Left: Eigenmodes ϕ_1 and ϕ_2 at $x = -20, 269$ and 480 km. Right: reconstructed solution for the vertical displacement δz, after interpolation on a finer grid. The topography is plotted in grey.

is general enough to encompass inhomogeneous stratification and background velocity. The solution was decomposed in terms of range-dependent normal modes and a set of ordinary differential equations, the coupled-mode equations. It is worth mentioning that, up to this stage, no approximation was made as far as the range-dependence of the medium properties are concerned. Therefore, it provides a useful tool for the investigation of the impacts of horizontal inhomogeneities on wave propagation. We then computed by an approximate solution that neglects the coupling between modes. Note that while the depth of the waveguide is constant in the present derivation, its variation can be taken into account using the same formulation. One interesting point of the method is obviously the low computational cost, compared to a complete linear resolution. Extension of the method to a three dimensional medium is unfortunately not straightforward, mainly because the equations for the vertical structure of the solution no longer lead to a standard eigenproblem. This work is currently under investigation.

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