The elephant in the room: how to define a rotation-aware Available Energy

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Abstract

The issue of how to include the effects of rotation in incompressible fluids within the Available Potential Energy framework is revisited. While it is trivial to show that the Coriolis force does not do any work on the system, by no means it can be ignored when considering the energetics of flows where rotation plays a $O(1)$ role (e.g., in large scale geophysical flows). In this short note, we sketch a framework to include such effects. The key point is to recognize that just as conservation of volume limits the minimum potential energy that a system can attain (because the mass cannot be all squeezed near the bottom), conservation of Potential Vorticity in general prevents the system from accessing certain states that would otherwise be accessible if volume (and hence, density) where the only conserved quantity along trajectories. The Casimirs emerge as the functions that encode in the local Available Energy (note we dropped the Potential qualifier) the dynamical constraints.

1 Introduction

The concept of Available Potential Energy (APE) in one form or another has been around since the early 1900's. Later, Shepherd (1993) showed that dynamically equivalent Hamiltonians can be introduced if the system admits Casimirs which can be used to define an APE. Without explicitly naming them Winters et al. (1995) used a specific class of Casimirs to construct a global APE and to study mixing within a stratified turbulent fluid. More recently,Scotti and White (2014), using the same Casimirs of Winters *et al.*, introduced a local APE. Leveraging the convexity of the local APE, they showed how it can be partitioned between a turbulent and a mean component, in the same way that the kinetic energy can be split between a mean and turbulent component. Zemskova et al. (2015) used this partition to calculate how KE and APE are split between mean and turbulent (eddy) component in a global ocean simulation. They showed that the amount of APE contained in the mean flow is 3 orders of magnitude larger than turbulent KE, mean KE and turbulent APE combined. Most of the mean APE is associated to the sloping isopycnals in the Southern Ocean. This suggests that, on planetary scales, APE must include the effects of rotation.

It is well known that the Coriolis force does not do any work on the system, therefore rotation "disappears" almost immediately when we move down the energetics path. In this paper, we reformulate the problem of APE using a variational approach that identifies the Casimirs with the Lagrange multipliers that enforce a set of constraints associated to the relevant conserved quantities. In non-rotating systems, conservation of density along Lagrangian trajectories is, with some caveats, the relevant constraint, and, by enforcing it, the framework recovers the local formulation of Scotti and White (2014). When rotation (strongly) affects the dynamics, in addition to conservation of density, we have to include conservation of Potential Vorticity (PV) among the constraints. In the latter case, the "ground state", i.e. the state that minimizes the total energy while preserving the initial distribution of density and PV, must contain more energy than in the case where the PV constraint is not enforced, and thus the Available Energy is less.

The paper follows a propaedeutic approach: we first discuss the main ideas with a toy model. Next, we apply them to the more interesting geophysical problem of rotating shallow-water flows, for which, aside from the total volume, PV is the dynamically relevant constraint; finally we formulate the problem for a rotating Bousinnesq fluid where both PV and density are relevant.

2 An illustrative example: a classical spin system in an external magnetic field

The precession of a classical spin under the action of a steady magnetic field provides the simplest yet non-trivial example of the type of analysis that we will pursue in the rest of the paper. The phase space is three-dimensional. Let $\mathbf{S} = (S^1, S^2, S^3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ be the components of the spin and magnetic field respectively. Then

$$\frac{dS^i}{dt} = -(\mathbf{B} \times \mathbf{S})^i. \tag{1}$$

Let

$$H(\mathbf{S}) = -S^k B_k \text{ and } J^{ij}(\mathbf{S}) = \epsilon^{ij}_k S^k, \tag{2}$$

be the Hamiltonian and the Poisson tensor (a function on the phase space), where ϵ^{ij}_k is the totally antisymmetric rank-3 tensor. Eq. (1) is equivalent to the Hamiltonian system

$$\frac{dS^i}{dt} = J^{ij}(\mathbf{S}) \frac{\partial H(\mathbf{S})}{\partial S^j}. \tag{3}$$

J being odd-dimensional *and* antisymmetric, we have $\det J = 0$, thus J is degenerate (for a discussion of Hamiltonian systems with singular Poisson tensors see Littlejohn, 1982, from which this example is adapted). This property has two important consequences: first, equilibrium points need not, and in general are not, extrema of the Hamiltonian function; second, it implies the existence of functionals $C(\mathbf{S})$ on the phase space such that

$$J^{ij} \frac{\partial C(\mathbf{S})}{\partial S^j} = 0, \tag{4}$$

everywhere in phase space (recall that the determinant is zero everywhere in phase space). For this particular simple system, it is immediate to verify that the functionals that satisfy (4) are arbitrary functions of $S_i S^i$, and without loss of generality (if C is a Casimir, then any $f(C)$, with f sufficiently smooth, is a Casimir too) we can take

$$C(\mathbf{S}) = a|\mathbf{S}|^2. \tag{5}$$

Since they commute with any functional over the phase space, functionals that satisfy (4) are called Casimirs, and are obviously conserved quantities. They owe their existence and nature to the degeneracy of the Poisson tensor, and not on the specific nature of the Hamiltonian. The existence of Casimirs means that some quantities are conserved along the motion (the Casimirs itself). Thus, Casimirs introduce a foliation of the phase space, that is the phase space can be partitioned in manifolds (leaves), within which orbits must be contained. Moreover, for a given J, the dynamics does not change if we replace H in (3) with

$$\mathcal{H}(\mathbf{S}) = H(\mathbf{S}) + C(\mathbf{S}). \tag{6}$$

Let us now approach the problem from a different direction: From (1) we know (by simple inspection, no need to introduce Casimirs) that the modulus of the spin $|S|$ does not change along trajectories, and we ask what is the minimum energy (as given by (2)) attainable by a point that at $t = 0$ has a spin whose modulus is s. This is a variational problem subject to a constrain, and we can turn into an unconstrained variational problem if we introduced a suitable Lagrange multiplier. That is, we seek to find the extrema of

$$\mathcal{H}'(\mathbf{S}) = H(\mathbf{S}) + \lambda(|S|^2 - |s|^2), \tag{7}$$

in the enlarged space $(\lambda, \mathbf{S})$. Without loss of generality we can set $\mathbf{B} = (0, 0, B)$, and the extrema are $(\lambda_e, \mathbf{S}_e) = \pm(B/2S_0, 0, 0, s)$, which corresponds to the minimum and maximum in energy. By comparing $\mathcal{H}$ with $\mathcal{H}'$, we see immediately that the part containing the Lagrange multiplier can be identified as the difference between a suitable Casimir evaluated on $\mathbf{S}$ and the same Casimir evaluated on the extremal solution. Further, the solution of the optimization determines not only the extrema, but also singles out the *right* Casimir which is (here we choose the ones that minimizes the energy)

$$C_e(\mathbf{S}) = \frac{B}{2s}|S|^2. \tag{8}$$

Note that the *right* Casimir depends on the spin of the system at $t = 0$, as well as the details of the Hamiltonian (via its dependency on B). Since dynamically we cannot distinguish between H and $\mathcal{H}$, we can use the latter as a definition of energy, so that the energy at the minimum is

$$\mathcal{H}(S_0) = -B_i S_0^i + C_e(\mathbf{S}_0), \tag{9}$$

and the Available Energy of a point on the leaf is

$$E_{AE}(S) \equiv H(\mathbf{S}) + C_e(\mathbf{S}) - \mathcal{H}(\mathbf{S_0}). \tag{10}$$

This is of course a very simple problem, but it highlights the main ingredients of the recipe applicable to a rotating fluid:

1. The degeneracy of the Poisson tensor implies the existence of conserved quantities *other* than the *naive* energy functional.

2. The same degeneracy results in equilibrium solutions that need not be extrema of the *naive* energy functional.

3. Point 1 implies that the dynamics is constrained to a manifold of the phase space. The constraints can be accounted by the Casimir. To each manifold its own Casimir.

4. For a *given set of constraints*, we can both determine the appropriate Casimir and find the state of minimum energy.

5. By adding the appropriate Casimir to the *naive* energy, we can introduce a dynamically equivalent energy functional (the Available Energy) so that the equilibrium solutions are its extrema.

6. The Available Energy measures the amount of energy that can be extracted from the system by a mechanism that does not change the constraints.

3 A first fluid example: rotating shallow-water flows

Now we consider the more interesting problem of a shallow water system observed in a rotating frame under f-plane approximation. In this case, the phase space is described by the horizontal components of the velocity vector $\mathbf{v}$ and the height of the free-surface h. Without loss of generality, we choose units in which the density is equal to one. This system admits a Hamiltonian formulation with a degenerate Poisson tensor (a clearly presented Hamiltonian formulation for several geophysically relevant flows, including this one, can be found in Shepherd, 1990). The Hamiltonian is

$$H[h, \mathbf{v}] = \int \left(h\frac{v^2}{2} + g\frac{h^2}{2} \right) dA, \tag{11}$$

where dA is the area element. Here and thereafter we use square brackets to indicate that a quantity is a functional of the argument(s). The Casimirs are

$$C[h, \mathbf{v}] = \int h\mathcal{F}(p) dA, \tag{12}$$

where $p = (\omega + f)/h$, the sum of the relative and planetary vorticity, is the PV, and $\mathcal{F}$ is an arbitrary function of PV. The integral is taken over the domain occupied by the fluid. Relatively to the simple spin system considered earlier, we have now a much wider latitude in choosing the Casimirs.

Consider the one-parameter set of functions $\mathcal{F}_s(p) = \theta(p-s)$, where $\theta(x)$, is the Heaviside function and a trajectory $q(t)$ in the phase space. The collection of the values spanned by the Casimirs associated to this family evaluated over $q(t)$ defines a function (h_q and p_q are the surface elevation and PV of q)

$$C(s, t) = \int h_{q(t)} \theta(p_{q(t)} - s) dA, \tag{13}$$

which has the straightforward interpretation of measuring the volume occupied by parcels (in the Lagrangian sense!) of fluid whose PV is greater than s at time t. As the flow evolves, $q(t)$ meanders in the phase space, but $dC/dt = 0$. As parcels move around, the volume occupied by parcels with PV greater than s cannot change. In particular, the constancy of $C(p_{\min}, t)$ expresses the conservation of total fluid volume. $C(s, t)$ evaluated at an arbitrary time, which, from now on, we can take to be 0, identifies the manifold in phase space to which the point $q(t)$ in phase space belongs. Also, from now on we drop the explicit dependence on time. We call $C(s)$ the PV Volumetric Distribution Function (VDF). Finally, let the support of $C(s)$ be $[p_{\mathrm{m}}, p_{\mathrm{M}}]$.

To calculate the Available Energy of a point q in phase space, we first calculate the corresponding PV VDF via (13). Next we set up the variational problem using a suitable Lagrange multiplier

$$\mathcal{H} = H + \int_{p_{\mathrm{m}}}^{p_{\mathrm{M}}} \psi(s) \left[\int h\theta(p - s) dV - C(s) \right] ds \tag{14}$$

where $\psi(s)$ is the Lagrange multiplier that enforces the PV VDF constraint. Let

$$\Psi(p) \equiv \int_{p_{\mathrm{m}}}^{p} \psi(s) ds. \tag{15}$$

Interchanging the order of integration, and using the properties of the Heaviside function, we obtain

$$\mathcal{H}[h, \mathbf{v}, \psi] = \int \left[h\frac{v^2}{2} + g\frac{h^2}{2} + h\Psi(p)\right] dA + \int_{p_{\mathrm{m}}}^{p_{\mathrm{M}}} \psi(s)C(s)ds, \tag{16}$$

which connects the Lagrange multiplier ψ to a suitable Casimir (we use Casimir to denote both the proper functional form, i.e., integrated over the domain or just the integrand). Taking the Frechèt derivatives we obtain[1]

$$0 = \frac{\delta\mathcal{H}}{\delta h} = gh + \frac{v^2}{2} + \Psi(p) - p\Psi'(p), \tag{17}$$

$$0 = \frac{\delta\mathcal{H}}{\delta v^i} = hv^i - \epsilon^{3ji}\partial_j\Psi'(p), \tag{18}$$

$$0 = \frac{\delta\mathcal{H}}{\delta\psi} = \int h\theta(p-s)dA - C(s). \tag{19}$$

We have four equations (17-19) whose solutions, denoted with an asterisk, $(h_*, \mathbf{v}_*, \Psi_*)$ specify the location of the extrema on the manifold identified by the given PV VDF as well as the Casimir of the manifold. Before considering solutions to the above system, we point out some general properties. We have already observed that conservation of total volume is already accounted for as the $s \to p_m$ limit of (19). Moreover, the system of equations is invariant under the map $(h, \mathbf{v}, \Psi) \to (h, \mathbf{v}, \Psi + \lambda p)$, and we leave to the reader to verify that the total vorticity in the ground state is determined by selecting the appropriate λ (hint: λ is Lagrange multiplier associated to conservation of the total (relative plus ambient) vorticity). From (18) we see that $\psi(p_1) - \psi(p_2)$ is the flow rate carried by the ground state between two streamlines having (constant) PV equal to p_1 and p_2 respectively. Equivalently, the Lagrange multiplier evaluated on the PV is the streamfunction for the transport.

Once a solution is determined, we modify the *naive* Hamiltonian by the addition of the manifold-dependent Casimir

$$H_c(h, \mathbf{v}) = h\frac{v^2}{2} + \frac{gh^2}{2} + h\Psi_*(p). \tag{20}$$

The total Available Energy for a point $(h, \mathbf{v})$ on the manifold is

$$\mathcal{E}_{\mathrm{AE}}[h, \mathbf{v}] \equiv \int [H_c(h, \mathbf{v}) - H_c(h_*, \mathbf{v}_*)]dA = \int [H(h, \mathbf{v}) - H(h_*, \mathbf{v}_*)]dA. \tag{21}$$

The last equality follows from the fact that if $(h, \mathbf{v})$ and $(h_*, \mathbf{v}_*)$ are on the same manifold, they have the same PV VDF whence $\int h\psi_*(p)dA = \int h_*\psi_*(p_*)dA$, a simple exercise that we leave to the reader (hint: on the manifold, the term added to the Hamiltonian in (14) is zero). The local Available Energy as

$$E_{\mathrm{AE}} \equiv (H_c(h, \mathbf{v}) - H_c(h_*, \mathbf{v}_*))h^{-1}, \tag{22}$$

which, in general, is different from $H(h, \mathbf{v}) - H(h_*, \mathbf{v}_*)$. By construction, it is locally convex near the extrema, and thus satisfies the requirements of a "good" energy functional.

[1] The cavalier attitude with the position of the indexes is due to the underlying assumption is that the metric of the plane is diagonal, so co- and contra-variant components are identical.

For lack of space, we are not considering specific solutions. However, we invite the reader to verify that for manifolds whose extrema are such that $\omega_*/f \ll 1$, and $v_*^2/gh_* \ll 1$, so that $p_* \simeq f/h_*$, to lowest order the Casimir does not depend on $C(s)$. We call these geostrophic manifolds, and it is possible to characterize which $C(s)$ describe geostrophic manifolds. What is clear is that E_{AE} is very different from $g(h-h_*)^2/2$, (with $h_* = V/A$, V being the total volume and A the total area), which is what we would have obtained had we sought extrema subject *only* to conservation of total volume. The PV constraint shifts the extrema in phase space, *and in general adds a non-zero amount of kinetic energy to the ground state.*

4 A meatier case: continuously stratified, rotating Boussinesq equations

For these systems, the two quantities conserved along Lagrangian trajectories are buoyancy $b \equiv g(\rho_0 - \rho)/\rho_0$ and PV $p \equiv \nabla \cdot (\mathbf{\Omega} b)$, where $\mathbf{\Omega}$ is the total (ambient + relative) vorticity. Note that the above defition of p differs from the standard one by an inconsequential constant factor. The Hamiltonian is

$$H = \int \left(\frac{1}{2} v^2 - bz \right) dV. \tag{23}$$

The Casimirs of the system are the functionals

$$\mathcal{C}[b,p] = \int C(b,p) dV, \tag{24}$$

with $C(b,p)$ an arbitrary function, which follows trivially from the Lagrangian invariance of b and PV.

A manifold in phase space is identified by the (constant on the manifold) VDF of PV and buoyancy (in this case, a two-parameter family of Casimirs)

$$V(q,s) = \int \theta(p(\mathbf{x}) - q)\theta(b(\mathbf{x}) - s) dV, \tag{25}$$

which measures the volume of all fluid parcels whose buoyancy is greater than s and PV greater than q. Following the established blueprint, we seek to minimize the Hamiltonian subject to a given VDF of buoyancy and PV. Let

$$\begin{aligned} \mathcal{H} = H + \int_{p_{min}}^{p_{max}} \int_{b_{min}}^{b_{max}} \psi(q,s) \left[\int \theta(p(\mathbf{x}) - q)\theta(b(\mathbf{x}) - s) dV - V(q,s) \right] dq ds = \\ = H + \int C(p,b) dV - \int_{p_{min}}^{p_{max}} \int_{b_{min}}^{b_{max}} \psi(q,s) V(q,s) dq ds, \\ C(q,b) \equiv \int_{p_{min}}^{p} \int_{b_{min}}^{b} \psi(q,s) dq ds \end{aligned} \tag{26}$$

and by taking Frèchet derivatives w.r.t. b, $\mathbf{v}$ and ψ, we arrive at the following set of equations

$$\frac{\delta \mathcal{H}}{\delta b} = -z + C_b - \nabla \cdot (C_p \mathbf{\Omega}) = 0, \tag{27}$$

$$\frac{\delta \mathcal{H}}{\delta v^i} = v^i + \nabla_i \times (C_p \nabla b) = 0, \tag{28}$$

$$\frac{\delta \mathcal{H}}{\delta \psi} = \int \theta(p(\mathbf{x}) - q)\theta(b(\mathbf{x}) - s) dV - V(q,s) = 0, \tag{29}$$

where $C_b = \partial C/\partial b$ and similarly for p.

4.1 Apedic manifolds

In our formulation, the Winters et al. (1995) APE, based on the isochoric resorted profile, corresponds to a ground state in which $\mathbf{v}_* = 0$ and $b_* = b_*(z)$. This ground state can be obtained from the system above if (29) is replaced by its $q \to -\infty$ limit, i.e. only enforcing the conservation of buoyancy constraint. In this case, assuming a simple container of constant area A, simple algebra shows that we can take $C_{*p} = 0$ (again, denoting with asterisks the solution), and thus $\mathbf{v}_* = 0$. Then (27) gives

$$z = C_{*b}(b), \tag{30}$$

hence b must be a function of the vertical coordinate alone. Assuming (30) can be inverted (which will have to be established a posteriori), we can use b as a vertical coordinate. Then, from the $q \to -\infty$ limit of (29) and (30) we have

$$\int \theta(b-s)\,|C_{*bb}|\,dbdxdy = V(-\infty, s). \tag{31}$$

Taking the derivative w.r.t. s of the above expression we obtain

$$|C_{*bb}| = -V_b(-\infty, b)/A. \tag{32}$$

Choosing the sign corresponding to the minimum, we recognize *ipso facto* $C_{*b}(b)$ as the reference height $z_*(b)$ based on an isochoric restratification introduced by Winters et al. (1995), and because of (30) its inverse gives the buoyancy profile $b_*(z)$ of the ground state. We leave to the reader to verify that if the VDF satisfies

$$\frac{V_b(p,b)}{V_b(-\infty,b)} = H(-f_z N_*^2(b) - p), \tag{33}$$

where $N_*^2(b) = \partial b_*/\partial z$ is the Brunt-Väisälä (BV) frequency of the reference state, the Winters et al. ground state is a solution to the original system as well. In particular, for a non-rotating case (33) implies that over manifolds where the PV is constant and equal to zero, the Winters *et al.* solution, and the localized theory of Scotti and White (2014) which is based on the Winters ground state, is appropriate. We call manifolds whose buoyancy distribution coincides with an isochoric restratification apedic [2] manifolds. The velocity of the ground state of apedic manifolds need not be zero. Several interesting cases fall are described by apedic manifolds, e.g. any non-rotating flow which is initially two-dimensional. More generally, we have the following result: *In an inertial frame, a necessary condition for a manifold to be apedic is that the vorticity of the ground state normal to surfaces of constant PV must be constant along streamlines.* The proof, based on exterior calculus, is too long to fit in this note.

4.2 The local Available Energy

Once the ground state and the Casimir of a manifold is found, the *local* Available Energy is

$$E_{AE} = \frac{1}{2}(v^2 - v_*^2) - (b - b_*)z + C_*(b,p) - C_*(b_*, p_*). \tag{34}$$

[2] From the Greek ἄπεδος, meaning level, flat.

Note that, in general,*the ground state has both potential and kinetic energy.* Also, and this is a general result, if (b, p) belongs to the manifold identified by C_*, then $C_*(b, p) - C_*(b_*, p_*)$ does not contribute to the volume integrated Available Energy.[3] This is why, *inter alia*, the local formulation of Scotti and White (2014) recovers the global formulation of Winters et al. (1995). For strongly rotating systems, we conjecture the existence of geostrophic manifolds, akin to the ones discussed in sec. 3, characterized by large values of the potential energy. This mechanism would show that the exceedingly large amount of APE found in the ECCO2 analysis is due to having the "wrong" ground state.

5 The role of diabatic processes

The role of Casimirs is to enforce the appropriate constraints on the system. Diabatic (mixing) processes change the constraints over time, and this is why the evolution of the (local) Casimirs along Lagrangian trajectories in the presence of mixing is key in characterizing the irreversible effects of diabatic processes on energetics. Within the Winters *et al.* APE framework, mixing always raises the energy of the ground state, and thus it decreases the AE of the system. On systems where PV is a dynamically important constraint, the situation is more complex. The PV constraint can keep a substantial amount of energy "locked up" (both as kinetic and potential) in the ground state. Mixing, by eroding the PV constraint, may free up some of the energy. A clarification of the relevant mechanisms would help understanding the role of mixing in strongly rotating flows.

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[3]See p. 5 for a sketch of the proof.