

Transmission of Internal Waves Generated by a Localized Surface Forcing

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Abstract

Localized surface forcings are very common in physical scenarios, for instance, in the case of a storm over an ocean that lasts for a limited time. We perform a theoretical study of the transmission of two-dimensional internal waves generated by a localized surface forcing through non-uniform density stratifications. We solve the two-dimensional Boussinesq internal wave equations using a weakly viscous linear model for a stratification that gradually changes in a finite transition layer. It is observed that time-localization of the forcing contributes to the disappearance of transmission peaks that would otherwise be present in the case of a harmonic forcing. We then conclude by demonstrating how this directly impacts the downward energy flux of the internal waves.

1 Introduction

Internal waves are propagating disturbances in a density stratified medium that is gravitationally stable. Linear internal waves excited by a forcing in a uniformly stratified fluid propagate at a fixed angle to the horizontal. One of the first experimental evidences of internal waves was found by Mowbray and Rarity (1967), who observed a St. Andrew's Cross pattern by oscillating a cylinder in a stratified fluid. Internal waves are observed readily in the ocean and the atmosphere as they are mediums that are naturally stratified. It has been shown by Alford (2003) that internal waves contribute to the redistribution of energy available for global ocean mixing and are thus important inputs for climate models.

Oceanic and atmospheric stratifications are highly non-uniform. One such example is the double-diffusive stratification structure that has been observed in the central Canada Basin of the Arctic ocean (Timmermans et al. (2008)). It is thus important to study the propagation of internal waves through non-uniform stratifications. Sutherland and Yewchuk (2004) theoretically computed the transmission coefficient for harmonic internal waves encountering a mixed region sandwiched between linearly stratified layers. This study was then extended by Mathur and Peacock (2009) in studying the propagation of internal wave beams through non-uniform stratifications. An interesting analogy between a Fabry-Perot interferometer and harmonic internal wave transmission was shown both theoretically and experimentally by Mathur and Peacock (2010). Most recently, a study by Ghaemsaidi et al. (2016) has shown that layered stratifications have a rich transmission behavior for internal waves that are harmonic and horizontally periodic. Multiple transmission peaks exist due to interactions between propagating and evanescent waves (Ghaemsaidi (2015)).

All the above mentioned studies consider harmonic internal waves. In contrast, we consider internal waves generated by a time-localized forcing, which is potentially very different because of a higher frequency content in the forcing. We discuss the mathematical

formulation of the problem in section (2), present the results in section (3) and conclude in section (4).

2 Mathematical Formulation

The dynamics of two-dimensional linear internal waves propagating in a viscous stratified fluid is given by (Sutherland (2010)):

$$(\nabla^2 w)_{tt} + N(z)^2 w_{xx} = \nu (\nabla^4 w)_t \quad (1)$$

w is the vertical velocity field and $N(z)$ is the buoyancy frequency that is given by $N(z) = \sqrt{(-g/\rho_0)d\bar{\rho}/dz}$. Here, ρ_0 is the characteristic density of the fluid, $\bar{\rho}$ is the background density, g is the gravitational acceleration and ν is the kinematic viscosity of the fluid. The physical system under consideration is shown in figure (1). The buoyancy frequency changes from a value of N_1 to N_2 through a transition layer of thickness Δ that lies a distance L below the top boundary.

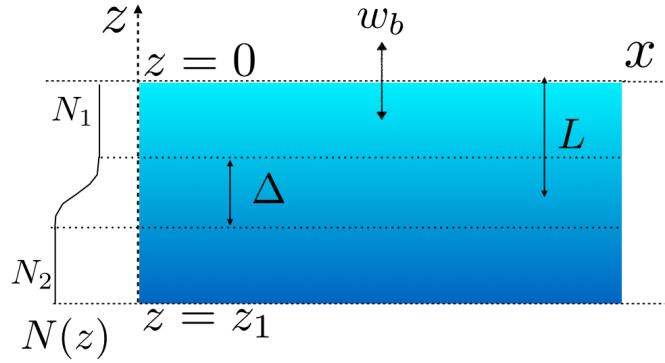


Figure 1: A sketch of the physical system. $w_b(x, t)$ is the forcing function that fixes the vertical velocity at the top boundary ($z=0$) of the system. The physical domain extends from $-\infty$ to $+\infty$ in the x direction. $z = z_1$ acts as a boundary through which waves are allowed to pass through freely. The thickness of the transition layer is Δ and its midpoint lies a distance L below the top of the boundary.

We solve equation (1) spectrally in x and t . A general vertical velocity forcing function can be decomposed into its Fourier modes as follows.

$$w_b(x, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{w}_b e^{i(kx - \omega t)} dk d\omega \quad (2)$$

For each Fourier mode of the forcing, we substitute an ansatz $\hat{w}(z) \exp(i(kx - \omega t))$ for $w(x, z, t)$ in equation (1) to obtain

$$\hat{w}_{zzzz} + \left(\frac{i\omega}{\nu} - 2k^2 \right) \hat{w}_{zz} + k^2 \left(k^2 + \frac{i\omega}{\nu} \left(\frac{N(z)^2}{\omega^2} - 1 \right) \right) \hat{w} = 0 \quad (3)$$

Equation (3) is solved using the MATLAB *bvp4c* solver to obtain $\hat{w}(z; \omega, k)$ (Ghaemsaidi et al. (2016)). The general solution at the top and bottom boundaries can be written as

$$\hat{w} = \begin{cases} \mathcal{I} \exp(m_1 z) + \mathcal{R} \exp(-m_1 z) & \text{at } z = 0 \\ \mathcal{T} \exp(m_2 z) & \text{at } z = z_1 \end{cases} \quad (4)$$

Here, \mathcal{I} , \mathcal{R} and \mathcal{T} are the amplitudes of the incident, reflected and transmitted waves through the transition layer. m_1 and m_2 are the local vertical wavenumbers in the N_1 and N_2 layers respectively, such that the group velocity of the incident and the transmitted waves points downwards and they are weakly damped. The representation in equation (4) allows us to implement two boundary conditions each at $z = 0$ and $z = z_1$. After obtaining the vertical structures for all the Fourier modes, we transform them back to the space-time domain to obtain the vertical velocity field $w(x, z, t)$.

3 Results

To demonstrate the effects of localization, we consider a velocity forcing function that is given by

$$w_b(x, t) = A \exp(i(k_0 x - \omega_0 t)) \exp\left(-\frac{(x - x_0)^2}{2\sigma_x^2}\right) \exp\left(-\frac{(t - t_0)^2}{2\sigma_t^2}\right) + c.c. \quad (5)$$

The above function is essentially a harmonic function in x and t but is localized by the respective Gaussian functions. This widens the frequency and wavenumber content around ω_0 and k_0 depending on the widths of the Gaussian functions given by σ_t and σ_x respectively. A is the maximum value of the forcing velocity.

The following stratification profile is considered for our model case study.

$$N(z) = \left(\frac{N_1 + N_2}{2}\right) + \left(\frac{N_1 - N_2}{2}\right) \tanh\left(\frac{z + L}{\Delta/6}\right) \quad (6)$$

A snapshot of the theoretically computed vertical velocity field is presented in a non-dimensional form in figure (2).

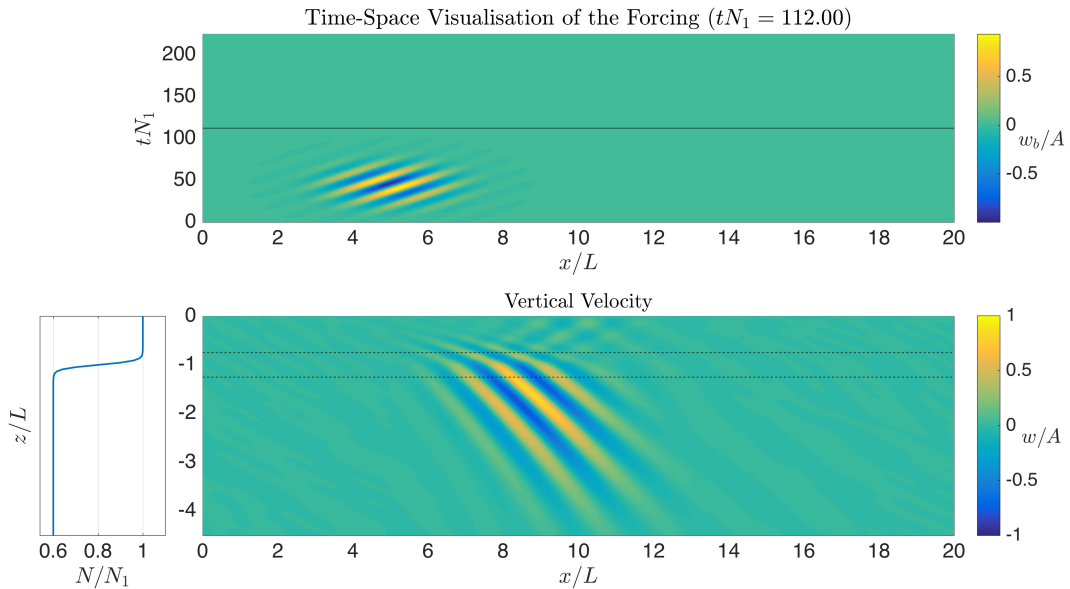


Figure 2: The velocity field at $tN_1 = 112$ (for $\Delta/L = 0.5$, $N_1/N_2 = 0.6$, $\omega_0/N_1 = 1/3$, $\sigma_t N_1 = 15$, $\sigma_x/L = 1$ and $k_0 L = 5$). The top panel shows the time-space nature of the forcing wherein the horizontal line indicates the current time. At this time instant, the forcing has already decayed. The bottom left panel indicates the stratification profile and the bottom right panel shows the vertical velocity field in the domain at the current time instant.

We now fix the non-dimensional parameters $\sigma_x/L = 1$ and $k_0L = 5$, and vary σ_t and ω_0 . From the obtained vertical velocity field, we compute the transmission parameter ($\tau_w = \max(w(z = z_1; x, t))/A$) and total transmitted energy (non-dimensionalized by $\rho_0 A^2 L^2$). These quantities are plotted in figure (3). For the lowest value of σ_t , both the quantities vary monotonically with ω_0 indicating the absence of interference effects. As the value of σ_t is increased, more time is available for the wave beam to reflect off the transition region and interfere with the forcing. This leads to the appearance of interference peaks.

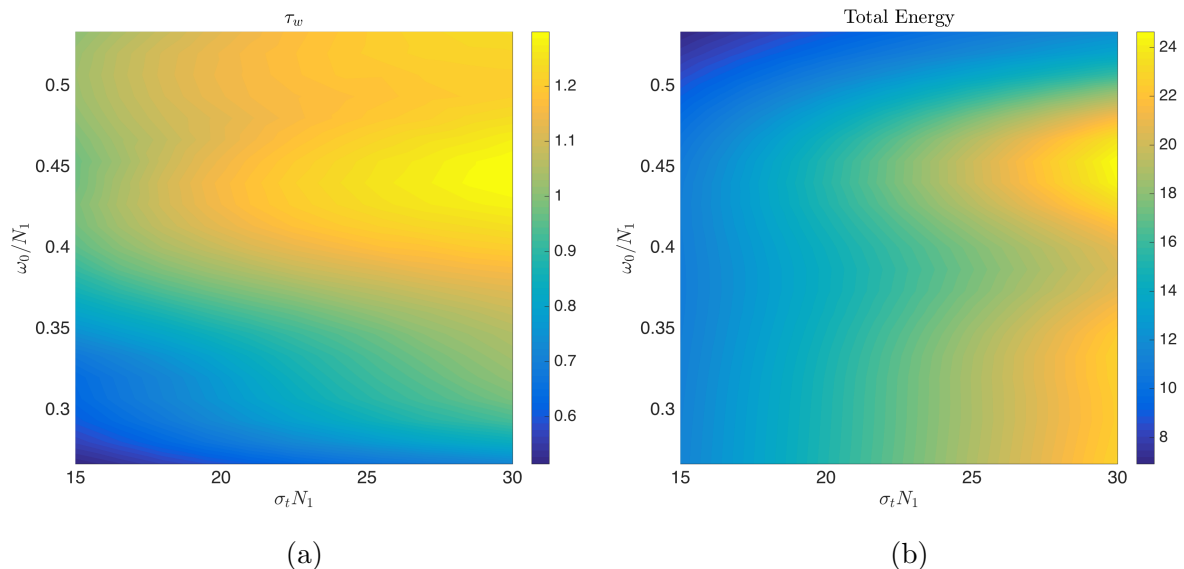


Figure 3: Plots of (a) transmission parameter based on w (τ_w) and (b) the total dimensionless energy

4 Conclusion

We have performed a theoretical study of the transmission of internal waves that are generated by a boundary forcing that is localized in time. This has been done by employing a weakly viscous pseudo-spectral model to solve the linear internal wave equations for any arbitrary stratification and surface forcing. As the forcing gets localized, the frequency content around the dominant frequency gets wider which affects the interference peaks. We choose a sample forcing function and perform a case study to demonstrate this effect. Through a parametric study, we show that as the forcing gets de-localized, we start observing the appearance of interference peaks in the transmission spectra. This in turn affects the total energy that is transferred due to the forcing.

Future work in this study would involve understanding the effect of localization in both space and time. Physical arguments pertaining to the group velocity of internal waves belonging to the dominant wavenumber and frequency can help in identifying the parameter regimes where a harmonic analysis can produce incorrect results.

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