Flows Induced by 1D, 2D and 3D Internal Gravity Wavepackets

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Abstract

Even at small amplitude, internal wavepackets induce a flow somewhat analogous to the Stokes drift induced by surface waves. The structure of this flow changes qualitatively depending upon whether the wavepacket is one-dimensional (horizontally periodic, vertically localized and spanwise-uniform), two-dimensional (along-wave and vertically localized, but spanwise-uniform), and three-dimensional (localized in all three spatial dimensions). Over the vertical span of a one-dimensional wavepacket, a positive flow is induced that translates with the vertical group velocity of the wavepacket. A two-dimensional wavepacket induces long internal waves that follow the wavepacket like a bow wake behind a ship. This was first shown by Bretherton (1969), who also showed that these waves disappear and a local circulation is induced if the wavepacket is three dimensional with comparable extent in all three dimensions. Using a quasi-monochromatic wavepacket approach, Bretherton's results are reproduced and extended to show that long waves as well as a local circulation are induced by three-dimensional wavepackets of relatively wide spanwise extent.

1 Introduction

Internal gravity waves move vertically through a continuously stratified fluid transporting momentum and irreversibly accelerating the background flow where they break. Even before breaking, however, localized internal wavepackets induce transient flows. In part, this is a consequence of the divergence of the momentum flux, which is zero far from the wavepacket and largest around its centre. The result is a "divergent-flux induced-flow" (DF) which scales as the amplitude squared of the waves. If this flow is itself divergent, then a response flow (RF) develops, driven for example through order amplitude-squared corrections to the pressure field. Together the sum of the divergent-flux and response flows give the total wave-induced mean flow.

For two- and three-dimensional wavepackets, the induced flow can extend far from the wavepacket itself. Thus momentum transport by the wavepacket is not necessarily localized to the wavepacket itself, as in the phenomena of long internal waves induced by two-dimensional wavepackets (van den Bremer and Sutherland (2014)) and "remote recoil" due to horizontally recirculating flows around three-dimensional wavepackets (Bühler and McIntyre (2003)). If moderately large amplitude, the induced flows can Doppler shift the waves in the wavepacket leading to modulationally stable or unstable evolution, which can be well-represented by a nonlinear Schrödinger equation for one-dimensional wavepackets (Akylas and Tabaei (2005); Sutherland (2006)). In particular, hydrostatic anelastic waves are modulationally stable and, as a result, overturn many density scale heights above the overturning level predicted by linear theory (Dosser and Sutherland (2011)).

Even without consideration of weakly nonlinear feedbacks, there remain several outstanding problems regarding the flows induced by internal wavepackets including the influence of horizontal boundaries upon vertical modes, Coriolis forces and aspect ratio of the amplitude envelope. This work will focus upon the influence of the last of these. Specifically, bridging the gap between two- and three-dimensional wavepackets we will examine flows induced by three-dimensional wavepackets of wide spanwise extent, as might be induced by stratified flow over mountain chains, as in the Rocky Mountains.

2 Theory

Here the theory for flows induced by 1D, 2D and 3D wavepackets are reviewed and then extended to the consideration of spanwise-wide wavepackets. For simplicity, the background flow is taken to be stationary, the stratification uniform and the Boussinesq approximation is invoked.

2.1 1D wavepackets

For a one-dimensional (horizontally periodic) wavepacket, the structure of the wave-induced flow follows directly from the horizontal momentum equation written in flux-form and averaged over one horizontal wavelength (Sutherland (2010)):

$$\partial_t \langle u \rangle = -\partial_z \langle uw \rangle \,, \tag{1}$$

in which angle-brackets denote averaging in x, and u and w represent the x- and z-velocity components, respectively. The velocities include both the flow directly attributed to the waves and also the wave-induced flow. Denoting the non-dimensional amplitude by α, the velocity of the waves is

$$u^{(1)} = \frac{1}{2}\alpha A_u(Z,T)e^{\imath(k_0 x + m_0 z - \omega_0 t)} + c.c. \quad \text{and} \quad w^{(1)} = \frac{1}{2}\alpha A_w(Z,T)e^{\imath(k_0 x + m_0 z - \omega_0 t)} + c.c., \tag{2}$$

in which $c.c.$ denotes the complex conjugate, $\vec{k}_0 = (k_0, m_0)$ is the wavenumber vector, ω_0 is the corresponding frequency, and A_u and A_w are the amplitude envelopes of the quasi-monochromatic wavepacket, which are related through the polarization relations by $A_w = -(k_0/m_0)A_u$. The amplitude envelopes themselves vary according to the slow space and time variables Z and T, respectively. Explicitly, $Z = \epsilon(z - c_{gz}t)$ and $T = \epsilon^2$, in which c_{gz} is the vertical group velocity and $\epsilon = 1/(k_0\sigma)$ is small assuming the vertical extent of the wavepacket, σ, is large compared to the wavelength. Weak dispersion dictates that the time-change of the wavepacket in a frame moving with the group velocity is of order ϵ^2. The magnitudes of A_u and A_w are of order unity with the nondimensional vertical displacement amplitude of the waves given by $\alpha = A_0 k_0$.

Equation (1) shows that the divergence of the momentum flux accelerates a flow whose magnitude goes as α^2. We denote this "divergent-flux induced-flow" by $\alpha^2 u_{\text{DF}}$. Just as the amplitude envelope translates vertically at the group velocity, u_{DF} is expected to be a function of the translating co-ordinate Z and slow-time variable T. Keeping leading-order terms of order $\alpha^2\epsilon$ in (1), we find $-c_{gz}\partial_Z u_{\text{DF}} = -\partial_Z(A_u A_w^\star)/2$, in which the star denotes the complex conjugate. Because u_{DF} is itself non-divergent it accounts entirely for the flow that is induced. Hence, using the polarization relations, the total induced flow by a one-dimensional wavepacket is given by

$$u^{(2)} = \alpha^2 u_{\rm DF} = \left\langle u^{(1)} w^{(1)} \right\rangle / c_{gz} = \frac{1}{2}\alpha^2 N_0^2 |\vec{k}_0| |A|^2, \tag{3}$$

in which $A \equiv A(Z, T)$ is the normalized amplitude of the vertical displacement field. The expression in (3) can also be derived from the pseudomomentum per unit mass, $-\langle \xi\zeta \rangle$, in which ξ is the vertical displacement and ζ is the spanwise vorticity associated with the waves (Scinocca and Shepherd (1992); Bühler (2009); Sutherland (2010)).

2.2 2D wavepackets

We now consider a wavepacket that is localized both in x and z, but has uniform spanwise extent. The amplitude envelope of the waves now depends upon the two slow spatial variables $X = \epsilon(x - c_{gx}t)$ and $Z = \epsilon(z - c_{gz}t)$. Considering only the influence of the advective terms in the horizontal and vertical momentum equations upon the acceleration gives

$$-\vec{c}_g \cdot \nabla(u_{\rm DF}) = -\nabla \cdot (A_{\vec{u}} A_u^\star) \quad \text{and} \quad -\vec{c}_g \cdot \nabla(w_{\rm DF}) = -\nabla \cdot (A_{\vec{u}} A_w^\star),$$

in which $A_{\vec{u}} = (A_u, A_w)$. It follows that

$$(u_{\rm DF}, w_{\rm DF}) = \frac{1}{2} N_0^2 |\vec{k}_0| |A|^2 (1, c_{gz}/c_{gx}). \tag{4}$$

Unlike the case of one-dimensional wavepackets, $(u_{\rm DF}, w_{\rm DF})$ is itself a divergent flow field. Thus the non-advective terms in the momentum equation at order α^2 must also come into play in order to ensure that the total induced flow is non-divergent.

To derive the total induced flow, the fully nonlinear equations of motion are combined into a single differential equation for the streamfunction, ψ:

$$\underbrace{\Big[\partial_{tt}(\partial_{xx} + \partial_{zz}) + N_0^2 \partial_{xx}\Big]}_{\equiv L} \psi = \underbrace{\nabla \cdot \Big[\partial_t\big(\zeta\, \vec{u}\big) + N_0^2 \partial_x\big(\xi\, \vec{u}\big)\Big]}_{\equiv \vec{F}}. \tag{5}$$

Simply substituting the structure of the waves, as in (2) with corresponding expressions for vorticity and vertical displacement, the slow variations of the nonlinear terms on the right-hand side of (5) give zero: this is ultimately a consequence of the fact that internal plane waves are an exact solution of the fully nonlinear equations of motion. The left-hand side then reveals that the differential equation for the induced flow at leading order in ϵ is $\partial_{xx}\psi_2^{(2)} = 0$, in which the subscript indicates that this term enters at order ϵ^2. Thus it is expected that the wavepacket should induce disturbances with long horizontal structure compared to the wavepacket width. Mathematically it means that the x-derivatives on the left-hand side of (5) are order ϵ smaller than the z- and t-derivatives.

In order to formulate the induced flow more accurately, it is necessary to consider the order ϵ corrections to the the wave structure, which accounts for the fact that the wavepacket is localized in the horizontal and vertical. Somewhat arbitrarily assuming that the vertical displacement field of the waves is given exactly by $\xi^{(1)} = \frac{1}{2}\alpha A(X, Z, T) e^{\imath\phi} + c.c.$ with $\phi \equiv k_0 x + m_0 z - \omega_0 t$, one finds for example that $w^{(1)} = -\imath\omega_0 A e^{\imath\phi} + [\epsilon_x c_{gx} A_X + \epsilon_z c_{gz} A_Z] e^{\imath\phi}$. Other values are tabulated explicitly in van den Bremer and Sutherland (2014).

Substituting these expressions into the right-hand side of (5), extracting the slow variations and knowing that the forcing results in a horizontally long response, we find that the leading order differential equation from (5) having non-zero forcing is (van den Bremer and Sutherland (2014))

$$\left(c_{gz}^2 \partial_{\tilde{z}\tilde{z}\tilde{z}\tilde{z}} + N_0^2 \partial_{\tilde{x}\tilde{x}}\right)\psi^{(2)} = \frac{1}{2}\alpha^2 N_0 |\vec{k}_0| c_{gz}^2 \partial_{\tilde{z}\tilde{z}\tilde{z}} |A|^2, \tag{6}$$

in which $(\tilde{x}, \tilde{z}) = \vec{x} - \vec{c}_g t = (X, Z)/\epsilon$ and the order-ϵ approximation $\partial_t \sim -c_{gz}\partial_{\tilde{z}}$ has been made.

The solution is found through Fourier transforms. In particular, using $u^{(2)} = -\partial_{\tilde{z}}\psi^{(2)}$ and $w^{(2)} = \partial_{\tilde{x}}\psi^{(2)}$, the Fourier amplitudes of the induced velocity are

$$\left(\widehat{u^{(2)}}, \widehat{w^{(2)}}\right) = \alpha^2 \left(\frac{c_{gz}^2 M^4}{c_{gz}^2 M^4 - N_0^2 K^2}, \frac{c_{gz}^2 K M^3}{c_{gz}^2 M^4 - N_0^2 K^2}\right) \widehat{u_{\text{DF}}}, \tag{7}$$

in which $\widehat{u_{\text{DF}}}(K, M) = (1/2) N_0 |\vec{k}_0| \widehat{|\mathcal{A}|^2}$ is the 2D Fourier transform of u_{DF}, given by (4), expressed in terms of the the transform of $|A|^2$. The special case of a 1D wavepacket corresponds to $\widehat{|\mathcal{A}|^2}$ being a delta function in K. It is obvious that (7) reduces in this case to the requirement that the induced vertical velocity is zero and $u^{(2)} = u_{\text{DF}}$, as expected.

The general result (7) can be inverse Fourier transformed and the integral in κ evaluated so as to ensure outward-propagating waves. In particular, the horizontal velocity field induced by a Gaussian wavepacket is (van den Bremer and Sutherland (2014))

$$u^{(2)}(\tilde{x}, \tilde{z}) = \frac{1}{8} c_{gz} N_0 |\vec{k}_0|\, A_0^2\, \sigma_x \sigma_z \int_0^\infty e^{-\mu^2 \tilde{z}^2/4} \mu^2 \sin\left(\mu\tilde{z} \;+\; \mu^2 \frac{c_{gz}|\tilde{x}|}{N_0}\right) d\mu. \tag{8}$$

This is plotted in Figure 1. The induced waves are the result of the wavepacket acting as a localized impulse that translates at the group velocity leaving behind the appearance of a bow wake. The vertical wavelength of the long waves scales as the vertical extent, σ_z of the wavepacket and the frequency is set by the condition that their vertical phase speed equals the vertical group velocity of the wavepacket.

It is important to emphasize that the induced waves have a horizontal extent orders of magnitude wider than the wavepacket itself. Thus, for example, a wavepacket generated by transient flow over a mountain range such as the Rocky Mountains can induce flows on the other side of the continent.

2.3 3D wavepackets with comparable horizontal extents

If a wavepacket is localized in all three spatial dimensions, then the localized forcing by the divergent-flux induced flow need not be compensated by long waves. Instead a response flow can be established that circulates horizontally around the wavepacket without the need to overcome buoyancy forces. This problem was first considered by Bretherton (1969) and is rederived here using the approach of quasi-monochromatic wavepackets.

The wavepacket is assumed to have leading-order structure given for the vertical displacement field by $\xi^{(1)} = \frac{1}{2}\alpha A(X, Y, Z, T) e^{\imath(k_0 x + m_0 z - \omega_0 t)} + c.c.$. As in the case of the 2D wavepacket, the divergence of the momentum flux results in the flow $(u_{\text{DF}}, 0, w_{\text{DF}})$ with u_{DF} and w_{DF} given by (4).

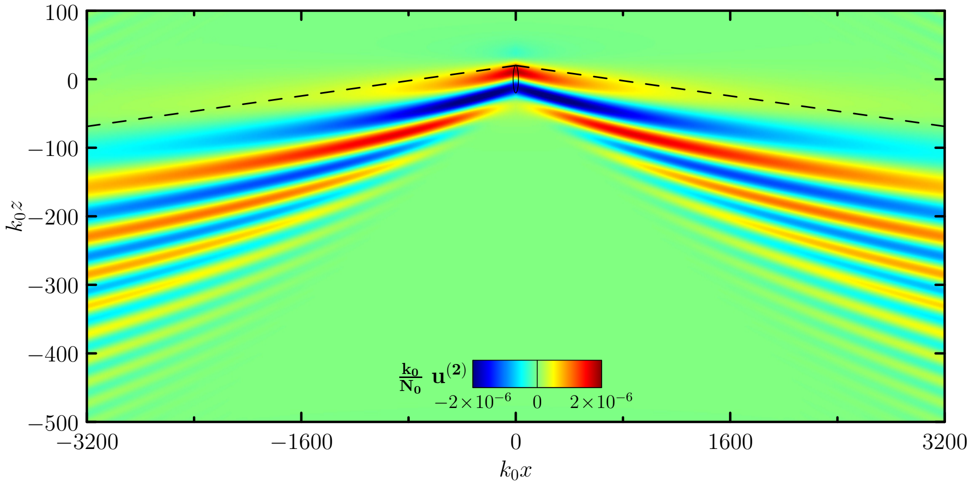

Figure 1: Horizontal velocity associated with flow induced by a Gaussian wavepacket centred at the origin with horizontal and vertical extent $\sigma_x = \sigma_z = 20k_0^{-1}$ (as indicated by ellipse in this high aspect ratio figure), and containing waves with $m_0 = -k_0$. The maximum vertical displacement amplitude $A_0 = 0.01k_0^{-1}$. The dashed lines indicates the predicted slope of the phase lines.

We seek a response flow $(u_{\rm RF}, v_{\rm RF}, w_{\rm RF})$ such that the total induced flow is purely horizontal. Hence $w_{\rm RF} = -w_{\rm DF}$. The total horizontal induced flow must be non-divergent. Hence $\partial_x u_{\rm RF} + \partial_y v_{\rm RF} = -\partial_x u_{\rm DF} = -\frac{1}{2}N_0|\vec{k}_0|\partial_x(|A|^2)$. Also assuming the response flow is irrotational, we have $\partial_x v_{\rm RF} - \partial_y u_{\rm RF} = 0$. The solution is found by Fourier transforming this pair of equations and solving separately for $\widehat{u_{\rm RF}}$ and $\widehat{v_{\rm RF}}$. Thus, for example, $\widehat{u_{\rm RF}} = -[K^2/(K^2+L^2)]\ \widehat{u_{\rm DF}}$. Adding this to the divergent-flux induced flow gives the Fourier transform in (K, L, M) space of the total induced flow:

$$\left(\widehat{u^{(2)}}, \widehat{v^{(2)}}\right) = \alpha^2 \left(\frac{L^2}{K_H^2}, \frac{-KL}{K_H^2}\right)\ \widehat{u_{\rm DF}}, \tag{9}$$

in which $K_H^2 = K^2 + L^2$.

Inverse Fourier transforming gives the flows shown, for example, in Figure 2. Unlike the case of 2D waves, here the induced flow extends horizontally only a few widths about the wavepacket.

2.4 3D wavepackets with wide lateral extent

Relaxing the condition that the induced flow is strictly horizontal, we derive a general formula for the wave-induced flow for a three-dimensional wavepacket following the analogous approach for 2D wavepackets. From the vector equation for vorticity and using incompressibility, the equations of motion can be recast as a linear operator on the velocity field being forced by a divergent field. Explicitly,

$$\underbrace{\begin{bmatrix} 0 & -\partial_t^2\partial_z & (\partial_t^2 + N_0^2)\partial_y \\ \partial_t^2\partial_z & 0 & -(\partial_t^2 + N_0^2)\partial_x \\ -\partial_t^2\partial_y & \partial_t^2\partial_x & 0 \end{bmatrix}}_{\equiv\ L} \vec{u} = \vec{F} \tag{10}$$

with

$$\vec{F} = \nabla\cdot\Big[-\partial_t\big(\vec{u}\otimes\vec{\zeta}\big) - N_0^2\big(\partial_x\big(\vec{u}\xi\,\hat{y} - \partial_y\big(\vec{u}\xi\,\hat{x}\big)\Big] + \partial_t\big(\vec{\zeta}\cdot\nabla\vec{u}\big). \tag{11}$$

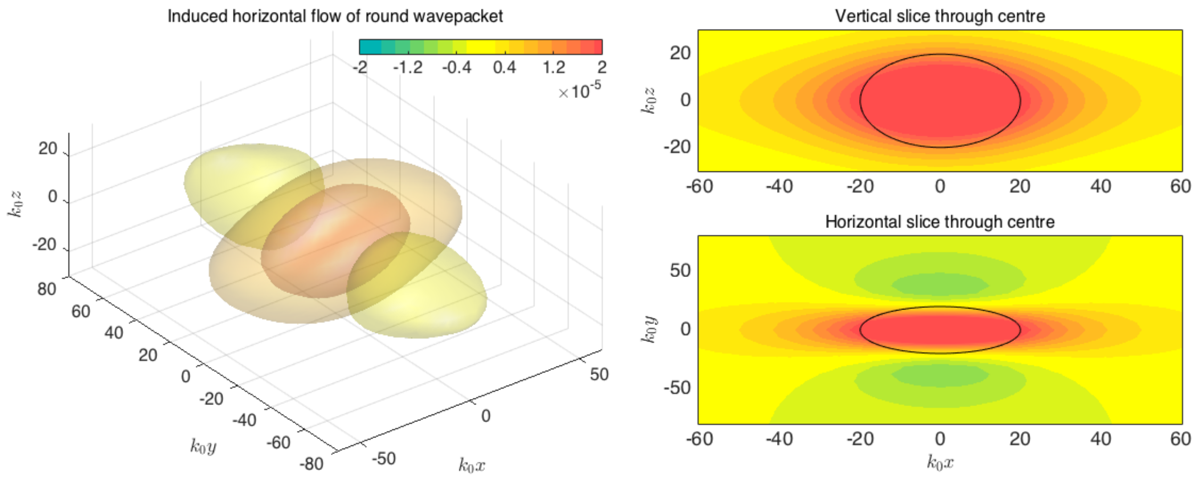

Figure 2: The x-component of velocity, $u^{(2)}$, associated with flow induced by a 'round' 3D Gaussian wavepacket centred at the origin with horizontal and vertical extents $\sigma_x = \sigma_y = \sigma_z = 20k_0^{-1}$ and containing waves with $m_0 = -k_0$. The maximum vertical displacement amplitude $A_0 = 0.01k_0^{-1}$.

As in the case of the 2D wavepacket, simple substitution of the expressions for the wavepacket at order α, neglecting the ϵ corrections gives zero on the right-hand side. A non-zero value for $\vec{F}$ is found by including the order ϵ corrections, as done for the 2D wavepacket. Fourier transforming both sides of (11), only keeping terms on the left-hand side at lowest order, gives the following for each component of the induced velocity

$$\left(\widehat{u^{(2)}}, \widehat{v^{(2)}}, \widehat{w^{(2)}}\right) = \alpha^2 \left(\frac{c_{gz}^2 M^4 - N_0^2 L^2}{c_{gz}^2 M^4 - N_0^2 K_H^2}, \frac{N_0^2 KL}{c_{gz}^2 M^4 - N_0^2 K_H^2}, \frac{-c_{gz}^2 K M^3}{c_{gz}^2 M^4 - N_0^2 K_H^2} \right) \widehat{u_{\mathrm{DF}}}. \quad (12)$$

This reduces to the formulae for 2D waves (7) by taking the spanwise wavenumber L to be zero (corresponding to $|A|^2$ being uniform in y). It reduces to the formula for compact 3D waves in (9) by taking $M = 0$, which can be interpreted as evaluating the horizontal flow at each height using the value of u_{DF} at that specific height. Thus we anticipate that a 3D wavepacket that has wide spanwise extent should induce both a local circulation and long waves. In particular, scaling analysis of the x-component of the induced velocity reveals that propagating waves should be substantial if the spanwise extent of the wavepacket is an order of magnitude larger than its vertical extent.

Figure 3 shows the flow induced by a wavepacket with lateral extent ten times wider than its vertical and along-wave extent. Being wider, the along-wave extent of the induced flow is correspondingly larger as is the lateral extent of the forward and recirculating flows. The magnitude of the flows is correspondingly smaller. But, while the recirculating flow decreases as the wavepacket width increases, the magnitude of the flow associated with radiating waves increases. The long waves are evident in the top-right plot of Figure 3 as flows with alternating sign below the recirculating flow.

Whether by waves or the recirculating flow, it is evident as in the case of 2D wavepackets, that a wide wavepacket induces a flow that is significant at distances ahead and behind the wavepacket that are orders of magnitude larger than the along-wave extent of the wavepacket itself.

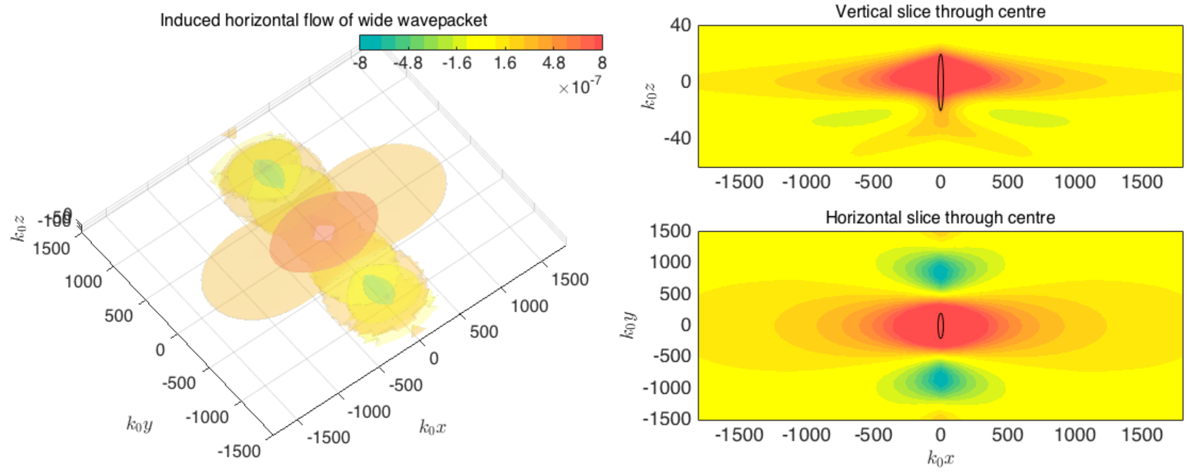

Figure 3: As in Fig. 2 but showing the flow induced by a wide 3D Gaussian wavepacket with lateral extent $\sigma_y = 200k_0^{-1}$ ten times larger than the vertical and along-wave extent.

3 Conclusions

We have examined the flows induced by Boussinesq internal wavepackets in uniformly stratified fluid. The general formula for 3D wavepackets is shown to reduce to formulae for 1D and 2D wavepackets if their amplitude envelopes are long in the along-wave and across-wave directions respectively. The recirculating flow predicted by Bretherton (1969) is reproduced for waves with spanwise extent comparable to the along-wave and vertical extent. But wide wavepackets also excite radiating waves, as predicted previously for 2D wavepackets of infinite lateral extent (van den Bremer and Sutherland (2014)).

Ongoing work is examining the influence of anelastic effects and Coriolis forces as well as flows induced by vertically confined internal modes. As shown for 1D wavepackets (Dosser and Sutherland (2011); Sutherland (2006)), these flows are expected to impact the evolution of the wavepacket itself if the waves have moderately large amplitude.

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