

On the mixing efficiency in stably stratified turbulence

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Abstract

The mixing efficiency R_f (more formally referred to as the flux Richardson number) is a widely used parameter that is intended to provide a measure of the amount of turbulent kinetic energy (TKE) that is irreversibly converted to background potential energy as a result of turbulent mixing in a stably stratified fluid. In this paper, three different definitions of the mixing efficiency that are commonly used in stratified turbulence are presented to highlight under what flow conditions they might be approximately equivalent and when such equivalences might break down. Analysis of two existing DNS datasets are presented to compare these various definitions. It was found that all three definitions were approximately equal when the gradient Richardson number $Ri_g \leq 1/4$. Conversely, when $Ri_g > 1/4$, significant variations are noticeable between the various definitions. Furthermore, the irreversible formulation of R_f is the only definition that is free from oscillations at higher Ri_g values.

1 Introduction

The quantitative characterization of diapycnal (irreversible) turbulent mixing in density stratified geophysical flows such as in the oceans and the atmosphere, remains a major challenge. This can be attributed primarily due to the complexity introduced into geophysical flows by density stratification in conjunction with complex topography and other physical phenomena (e.g. internal waves) associated with such flows. Regardless, accurate quantification of both momentum and diapycnal mixing of density is imperative given its importance for many practical applications such as air quality prediction, nutrient transport in water bodies and ocean circulation etc. It is common practice to quantify turbulent mixing in such flows using a turbulent (eddy) viscosity ν_t for momentum and a turbulent (eddy) diffusivity K_ρ for density, which are based on the gradient-diffusion hypothesis (Pope 2000). For a unidirectional shear flow, ν_t and K_ρ are defined as

$$\begin{aligned}\nu_t &= -\frac{\overline{u'w'}}{d\overline{U}/dz}, \\ K_\rho &= -\frac{\overline{\rho'w'}}{d\overline{\rho}/dz},\end{aligned}\tag{1}$$

where, \overline{U} is the mean streamwise velocity, $\overline{\rho}$ is the fluid mean density, $\overline{u'w'}$ is the Reynolds stress (turbulent momentum flux), $\overline{\rho'w'}$ is the turbulent density flux and an overbar $(\overline{\quad})$ implies averaging (spatial or temporal). Direct estimates of ν_t and K_ρ are hard to obtain from field measurements due inherent difficulties associated with estimating turbulent fluxes such as spatial resolution issues (e.g. in oceanic flows) as well as the coexistence of

internal wave motions and turbulence in stably stratified flows. A clean separation of flux contributions from turbulence to that from linear internal waves (which do not contribute to diapycnal mixing) remains challenging. Such difficulties have popularized the use of a number of indirect approaches that are frequently used in oceanography for the inference of momentum and heat fluxes, of which the two most commonly used approaches are those due to Osborn (1980) and Osborn and Cox (1972).

Major inherent assumptions of indirect methods are that the turbulent flow is statistically stationary and homogeneous. These assumptions are used to simplify the energetics of the turbulent flow field. Consider for example the Osborn (1980) model where (under these assumptions), the turbulent kinetic energy (TKE) equation can be simplified to obtain the diapycnal diffusivity of density as

$$K_\rho = \left(\frac{R_f}{1 - R_f} \right) \frac{\epsilon}{N^2}. \quad (2)$$

Here, ϵ is the dissipation rate of k , $N = \sqrt{-(g/\rho_0)\partial\bar{\rho}/\partial z}$ is the buoyancy frequency of the background (stable) density field and R_f , the mixing efficiency (also known as flux Richardson number) which provides a measure of the amount of TKE that is irreversibly converted to background potential energy, is traditionally defined as

$$R_f = \frac{B}{P}. \quad (3)$$

In equation (3), $B = g/\rho_0(\overline{\rho'w'})$ is the buoyancy flux and $P = -\overline{u'w'}(d\bar{U}/dz)$ is the rate of production of k . A canonical value of $R_f \approx 0.17$ was used by Osborn based off of some controlled laboratory experiments by Britter (1974), but many studies over the last few decades have attempted to parameterize the dependence of R_f on the strength of the stratification, which is commonly quantified in terms of the gradient Richardson number $Ri_g = N^2/S^2$, where $S = d\bar{U}/dz$ is the mean shear rate (Fernando 1991, Gregg 1987, Itsweire *et al.* 1993, Ivey and Imberger 1991, among others). However, a universal parameterization of R_f remains elusive due to both the ambiguities associated with single parameter formulations and the general complexity of stably stratified turbulence (Mater and Venayagamoorthy 2014).

Given the fact that the inherent assumptions of statistical stationarity and homogeneity are not always applicable in practice, be it in direct numerical simulations (DNS) or observational studies of geophysical flows, Ivey and Imberger (1991) proposed an alternative definition of R_f (hereafter denoted by R_f^{II}) as

$$R_f^{II} = \frac{B}{m} = \frac{B}{B + \epsilon}, \quad (4)$$

where the denominator m , in addition to P , includes the non-local (inhomogeneous) transport and unsteadiness terms. This definition therefore precludes the need to assume that the turbulence is stationary and/or homogeneous. However, it also suffers from the effects of countergradient fluxes that are common in strongly stratified flows. Thus, a third definition of R_f (denoted commonly as R_f^*) has been defined (Peltier & Caulfield 2003, Venayagamoorthy & Stretch 2010) where only the irreversible components of the buoyancy flux (i.e. the dissipation rate ϵ_{PE} of the turbulent (available) potential energy

E'_{PE}) and of the production term (i.e. the total dissipation rate, $\epsilon + \epsilon_{PE}$) are used as follows

$$R_f^* = \frac{\epsilon_{PE}}{\epsilon + \epsilon_{PE}}. \quad (5)$$

Here $\epsilon_{PE} = N^2 \epsilon_\rho (\overline{d\bar{\rho}/dz})^{-2}$ in which $\epsilon_\rho = \kappa \overline{(\nabla \rho')^2}$ is the dissipation rate of density (scalar) variance with κ defined as the molecular diffusivity of density. A conceptual explanation for the basis of this definition was provided by Venayagamoorthy & Stretch (2010). It is worth noting that for stationary homogeneous flows, $R_f^* = R_f^{II} = R_f$. As stated earlier, such conditions are rarely applicable in practice and hence it is imperative to investigate the differences and similarities between these three commonly used definitions of the flux Richardson numbers. This simple but important goal is the focus of this work.

2 The Data

Two existing DNS datasets that have all the necessary variables for computing R_f , R_f^{II} and R_f^* , as functions of the gradient Richardson number Ri_g are used in this study. The first dataset is from the DNS study of homogeneous shear flows by Shih *et al.* (2005, hereafter SKIF). These simulations are for temporally developing homogeneous turbulence subjected to different values of uniform mean shear rate and uniform stable stratification. The gradient Richardson numbers used in the simulations were $0.05 \leq Ri_g \leq 1$. For low Ri_g , the turbulent kinetic energy k grows in time, while at high Ri (> 0.25), k decays. All statistics were obtained by volume averaging over the computational domain. The data presented are for nondimensional shear time $St \geq 4$ to the end of the simulation time, in order to filter out the initial transients during the development phase of the turbulence encountered in the initialization of the simulations. Further details of the simulations can be found in SKIF. The second dataset is from the DNS study of stably stratified turbulent channel flow by García-Villalba and del Álamo (2011, hereafter GVA). For the purpose of this study, we use data from simulations performed at a friction Reynolds number of $Re_\tau = u_\tau \delta / \nu = 550$, with an initial stratification given by friction Richardson numbers of $Ri_\tau = |\Delta\rho|g\delta/\rho_0 u_\tau^2 = 60$. Here, u_τ is the friction velocity, δ is half of the channel depth and ν is the kinematic (molecular) viscosity and $|\Delta\rho|$ is the initial density difference between the bottom of the channel ($z = 0$) and the free-stream ($z = \delta$).

3 Results

The flux Richardson numbers R_f , R_f^{II} and R_f^* as defined by equations (3), (4) and (5), respectively, as functions of the gradient Richardson number Ri_g using the DNS data of SKIF and GVA, are shown in figure 1(a) and (b). There are two main points worth noting from figure 1(a). First for low Ri_g (up to $Ri_g \simeq 0.25$), it can be seen that all three definitions are approximately equivalent (albeit with some differences which on average are within $\pm 25\%$). The data suggests that the flux Richardson number (regardless of which definition is used) increases with Ri_g in a quasi-linear manner for small Ri_g ($\lesssim 0.1$) and continues to increase with decreasing slope for $0.1 \lesssim Ri_g \leq 0.25$. The favorable comparison between the three different definitions is encouraging in the sense that it highlights the fact that both buoyancy and momentum fluxes are dominated by turbulent processes in this shear dominated flow regime. The good agreement suggests that any of the three definitions could be used for inferring the irreversible mixing in the shear dominated regime in a stably stratified flow with a reasonable degree of accuracy.

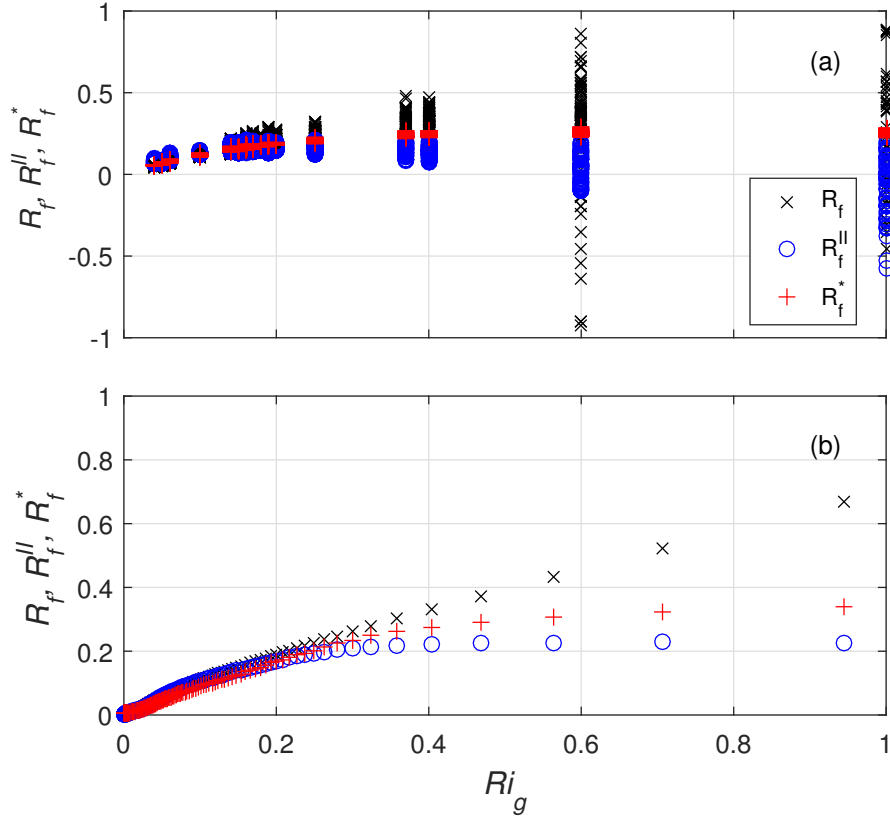


Figure 1: Comparison of flux Richardson numbers R_f , R_f^{II} and R_f^* as functions of the gradient Richardson number Ri_g , computed from (a) the homogeneous shear flow DNS data of Shih *et al.* (2005) and (b) the channel flow DNS data of García-Villaba and del Álamo (2011).

Second, as can be seen in figure 1(a), both R_f and R_f^{II} exhibit significant variability at higher Ri_g (> 0.25) while R_f^* shows almost negligible variations. Note that for a given Ri_g , the variability in R_f indicates oscillations in time of both the buoyancy flux and momentum flux, while for R_f^{II} , the variability signifies oscillations of the buoyancy flux. Physically, at high Ri_g (i.e. the so called buoyancy dominated flow regime), the production P of k becomes small while the buoyancy flux B is increasingly dominated by adiabatic displacements from linear internal waves. Hence, the traditional definition of R_f becomes less meaningful in such buoyancy dominated flow regimes. Similarly, R_f^{II} suffers from significant oscillations at high Ri_g . However, it is worth noting that unlike the trend in R_f , on average, R_f^{II} appears to decrease for $Ri_g \geq 0.25$ but becomes quickly less meaningful at higher Ri_g values (i.e. attains negative values) when significant countergradient buoyancy fluxes become prevalent. On the other hand, the irreversible flux Richardson R_f^* does not suffer from such issues since by definition, it excludes the effects of reversible contributions. Furthermore, R_f^* appears to asymptote to an approximate constant for high Ri_g .

Figure 1(b) shows a comparison of R_f , R_f^{II} and R_f^* using the DNS data of GVA for a fully developed stably stratified turbulent channel flow. Again, the relatively good agreement between the three definitions can be seen for $Ri_g \lesssim 0.25$. Given the important fundamental differences between the two sets of simulations i.e. the GVA simulations are for fully developed wall-bounded turbulent flow while the SKIF simulations are for time evolving homogeneous shear flows, it is encouraging to see similar trends in the behavior of all three definitions. This highlights the fact that the basic physics associated with shear

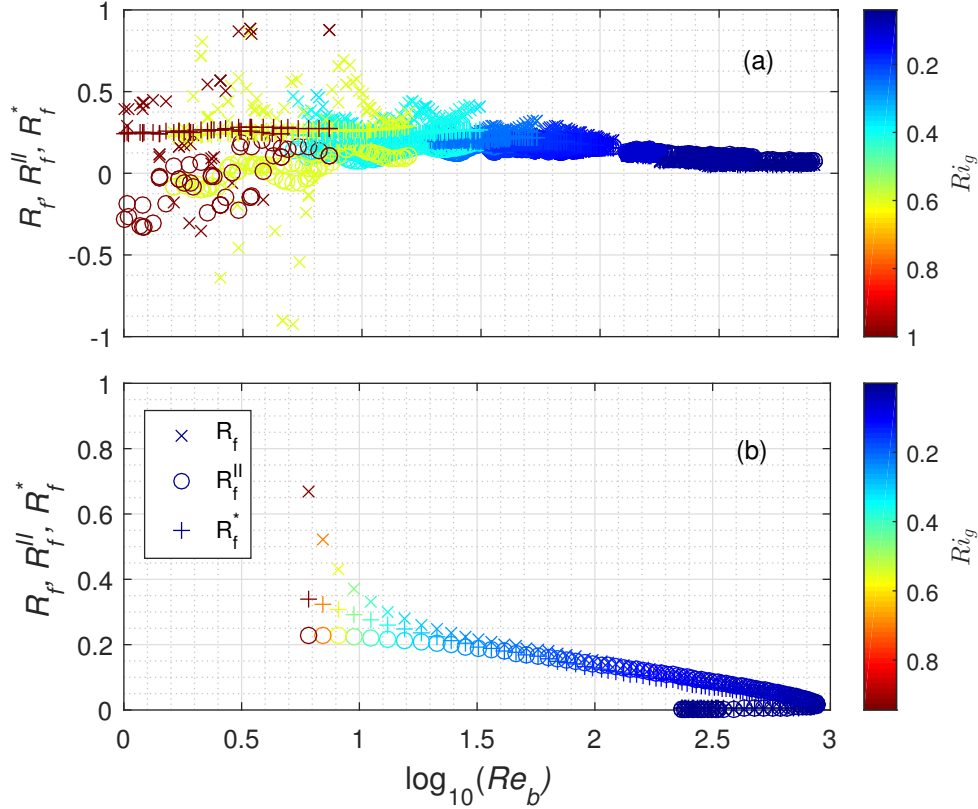


Figure 2: Comparison of flux Richardson numbers R_f , R_f^{II} and R_f^* as functions of the buoyancy Reynolds number Re_b , computed from (a) the homogeneous shear flow DNS data of Shih *et al.* (2005) and (b) the channel flow DNS data of García-Villaba and del Álamo (2011). The color bars show the corresponding gradient Richardson numbers Ri_g .

generated turbulence are at play in both of these flows as long as $Ri_g \lesssim 0.25$ and indicates that the fluxes are predominantly turbulent for both flows. On the other hand, for high Ri_g , the agreement between the three definitions decreases rather rapidly (but without oscillations since the data shown is for steady fully developed flow) especially between the traditional definition R_f and the other two definitions. This is due to the fact that the production P of k decays as the mean shear rate drops further away from the wall (i.e. in the inner core of the channel). On the other hand, in this far-wall region, reversible effects from linear internal waves are dominant and contaminate the buoyancy flux. Both R_f^{II} and R_f^* appear to level out with increasing Ri_g with R_f^{II} somewhat smaller than R_f^* , consistent with the trend also seen in figure 1(a). These results highlight the importance of separating out the reversible contributions to both the momentum and scalar fluxes in such time evolving (i.e. locally nonequilibrium) flows, especially in the strongly stratified buoyancy dominated flow regime.

The buoyancy Reynolds number $Re_b = \epsilon/(\nu N^2)$, where ν is the kinematic viscosity of the fluid, is widely used for parameterizing mixing in stratified turbulence (e.g. Shih *et al.* 2005). It is therefore instructive to explore the variation of R_f , R_f^{II} and R_f^* with Re_b . Figure 2(a) and (b) show the dependence of R_f , R_f^{II} and R_f^* with Re_b . It can be seen that all three definitions are in approximately good agreement for $Re_b \gtrsim 30$. This is also consistent with the low Ri_g regime ($\lesssim 0.25$) as can be seen in the color bar inserts in figure 2 and the dependence on Ri_g shown in figure 1. Moreover, at lower Re_b values

(< 30, and correspondingly higher Ri_g values), the agreement deteriorates similar to that seen in figure 1 with Ri_g . This figure underscores the fact that turbulence is suppressed (at least in these simulations) at higher Ri_g as buoyancy effects become dominant.

4 Conclusions

In this study, a careful comparison of the three definitions of the mixing efficiency R_f , R_f^{II} and R_f^* , that are commonly used to quantify diapycnal mixing in stably stratified flows was presented. Using DNS datasets of time evolving stably stratified homogeneous shear flow and turbulent channel flow, it was found that in the shear dominated flow regime ($0 \leq Ri_g \lesssim 0.25$), all three definitions are approximately equivalent. This is a key result in that this allows for the estimation of irreversible mixing directly from flux measurements, and vice versa for inferring fluxes from indirect estimates of dissipation rates. However, our analysis shows that the agreement between the three definitions deteriorates quickly as Ri_g increases above 0.25 with significant oscillations in both R_f and R_f^{II} in time evolving homogeneous stably stratified shear flows. Both R_f and R_f^{II} do not separate out the effects of countergradient fluxes that are pervasive in the buoyancy dominated flow regime. The irreversible flux Richardson number R_f^* is the only formulation that is free from such large oscillations and exhibits a clear positive definite trend with increasing Ri_g .

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