Understanding inertial instability on the f-plane with complete Coriolis force

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Abstract

We present analytical results allowing to understand the influence of usually neglected vertical component of the Coriolis force and of the vertical velocity contribution in its horizontal component upon wave-trapping in shear flows in stratified fluid, and upon related inertial instability.

1 Introduction

It is known (e.g. Gerkema and Shrira (2005)) that inclusion of the so-called non-traditional (NT) terms, i.e. of the usually neglected vertical component of the Coriolis force and of the vertical velocity contribution to its horizontal component, in the primitive equations for the atmosphere and the ocean changes the properties of near-inertial waves. In the recent paper (Tort et al., 2016) it was shown, on the basis of numerical solutions of the linearized primitive equations, that properties of sub-inertial waves trapped in the velocity shear change as well. As trapped waves are at the origin of the inertial instability (Zeitlin, 2008), the characteristics of the latter change too, with increasing growth rates and shifting instability thresholds, cf. Tort et al. (2016). In the present contribution, without having recourse to numerical analysis, we give analytic arguments allowing to understand, qualitatively and quantitatively, the changes induced by the NT effects in trapped modes and inertial instability. We first analyze in sect. 2 the NT inertial instability in the two-layer rotating shallow water model, following the lines of analogous treatment of traditional inertial instability by Zeitlin (2008), and then consider continuously stratified shear flows, following the lines of Plougonven and Zeitlin (2009), in sect 3. Section 4 contains conclusions and a discussion.

2 Inertial instability in two-layer rotating shallow water model with a rigid lid on the non-traditional f- plane

Our starting point is non-dissipative primitive equations in the Boussinesq approximation on the tangent f-plane with full Coriolis force

$$
\begin{aligned}
(\partial_t + u\partial_x + v\partial_y + w\partial_z)\, u - fv + Fw + \partial_x\Phi &= 0, &(1)\\
(\partial_t + u\partial_x + v\partial_y + w\partial_z)\, v + fu + \partial_y\Phi &= 0, &(2)\\
(\partial_t + u\partial_x + v\partial_y + w\partial_z)\, \rho &= 0, &(3)\\
\delta_{\mathrm{NH}}\, (\partial_t + u\partial_x + v\partial_y + w\partial_z)\, w + \partial_z\Phi + b - Fu &= 0, &(4)\\
\partial_x u + \partial_y v + \partial_z w &= 0, &(5)
\end{aligned}
$$

where x and y are zonal and meridional coordinates, u and v are corresponding components of velocity, $\rho = \rho_0 + \sigma(x, y, z, t)$ is density and $b = g\frac{\sigma}{\rho_0}$ is buoyancy. $\Phi = \frac{P}{\rho_0}$ is geopotential, constructed from pressure P and background density ρ_0. F is the "non-traditional" contribution to the Coriolis parameter, while the "traditional" Coriolis parameter is f. In terms of Earth's angular velocity Ω and latitude ϕ, $f = 2\Omega \sin\phi$ and $F = 2\Omega\cos\phi$. For convenience, following Tort et al. (2016), we included a switch $\delta_{\rm NH}$ which controls the (quasi-)hydrostatic approximation. It should be emphasized that the NT terms affect the hydrostatic balance, which thus becomes "quasi-hydrostatic" (White and Bromley, 1995).

2.1 A sketch of the derivation of the two-layer model

Following Zeitlin (2007) we vertically integrate the horizontal momentum and continuity equations (1), (2), (5) between two pairs of material surfaces: (z_1, z_2) and (z_2, z_3), supposing uniform densities $\rho_{1,2}$ in respective layers and taking $z_1 = 0$ and $z_3 = \text{const}$, and use the quasi-hydrostic relation resulting from (4) at $\delta_{\rm NH} = 0$ in order to express geopotential/pressure inside the layers in terms of the pressure at z_3 and vertical position. We arrive in this way to the following system of equations for the velocities $u_{1,2}$, $v_{1,2}$ and thicknesses $h_{1,2}$ of the respective layers:

$$\frac{d_2u_2}{dt} - fv_2 - F\left(\frac{1}{2}\partial_y v_2 h_2 - \partial_x(u_2h_2)\right) = -\frac{1}{\rho_2}\partial_x\pi, \tag{6}$$

$$\frac{d_2v_2}{dt} + fu_2 - \frac{F}{2}\partial_y u_2 h_2 = -\frac{1}{\rho_2}\partial_y\pi, \tag{7}$$

$$\frac{d_1u_1}{dt} - fv_1 - \frac{F}{2}\left(\partial_y v_1 h_1 + \partial_x\left(h_2u_1\right)\right) = -\frac{1}{\rho_1}\partial_x\pi - g\frac{\rho_1-\rho_2}{\rho_1}\partial_x h_1 + F\frac{\rho_2}{\rho_1}\partial_x\left(h_2u_2\right) \tag{8}$$

$$\frac{d_1v_1}{dt} + fu_1 - \frac{F}{2}\left(\partial_y(u_1h_1) + \partial_y h_1 u_1\right) = -\frac{1}{\rho_1}\partial_y\pi - g\frac{\rho_1-\rho_2}{\rho_1}\partial_y h_1 + F\frac{\rho_2}{\rho_1}\partial_y\left(h_2u_2\right) \tag{9}$$

$$\partial_t h_{1,2} + \partial_x\left(u_{1,2}h_{1,2}\right) + \partial_y\left(v_{1,2}h_{1,2}\right) = 0, \quad h_1 + h_2 = H_0 = \text{const}. \tag{10}$$

Here $\frac{d_{1,2}}{dt} = \partial_t + u_{1,2}\partial_x + v_{1,2}\partial_y$ are Lagrangian derivatives in respective layers, and π denotes the pressure under the rigid lid at $z = z_3$. These equations are the rigid-lid analog, which is much simpler, of the two-layer rotating shallow water equations with full Coriolis force and free upper surface derived by Stewart and Dellar (2010) and used by Tort et al. (2016).

2.2 Background flow and 1.5 dimensional reduction

As follows from (7), (9), stationary zonal flow solutions obey the following equations, which reflect modifications of the geostrophic balnce by NT terms:

$$fu_2 = -\frac{1}{\rho_2}\partial_x\pi + \frac{F}{2}\partial_y u_2(H - h_1), \tag{11}$$

$$fu_1 = -\frac{1}{\rho_1}\partial_y\pi - g\frac{\rho_1-\rho_2}{\rho_1}\partial_y h_1 + F\left[\frac{h_1\partial_y u_1}{2} + \partial_y h_1 u_1 + \frac{\rho_2}{\rho_1}\partial_y\left((H-h_1)u_2\right)\right]. \tag{12}$$

We are interested in the influence of the NT effects upon inertial instability, which was shown to be stronger at weak stratifications (Tort et al., 2016). Therefore, we will limit ourselves by frequently used in oceanography approximation where two densities are close $\rho_1 \approx \rho_2$, and their difference is significant only in the reduced gravity term $g\frac{\rho_1-\rho_2}{\rho_1} = g'$. We will be considering in what follows symmetric with respect to translations in x

configurations, like the stationary flows (11), (12). The translational symmetry leads to the "1.5 dimensional" reduction of (6) - (10):

$$
\begin{aligned}
\frac{d_2 u_2}{dt} - f v_2 - \frac{F}{2}\partial_y v_2 h_2 &= 0, && (13)\\
\frac{d_2 v_2}{dt} + f u_2 - \frac{F}{2}\partial_y u_2 h_2 &= -\frac{1}{\rho_2}\partial_y \pi, && (14)\\
\frac{d_1 u_1}{dt} - f v_1 - \frac{F}{2}\partial_y v_1 h_1 &= 0, && (15)\\
\frac{d_1 v_1}{dt} + f u_1 - \frac{F}{2}\left(\partial_y (u_1 h_1) + \partial_y h_1 u_1\right) &= -\frac{1}{\rho_1}\partial_y \pi - g\frac{\rho_1 - \rho_2}{\rho_1}\partial_y h_1 + F\frac{\rho_2}{\rho_1}\partial_y\left(h_2 u_2\right), && (16)\\
\partial_t h_{1,2} + \partial_y\left(v_{1,2} h_{1,2}\right) &= 0, \quad h_1 + h_2 = H_0 = \text{const.} && (17)
\end{aligned}
$$

2.3 Linearized equations and resulting eigenfrequency problem

Linearization of the 1.5 dimensional equations (13) - (17) about "barotropic" solutions with $h_1 = H_1 = \text{const}$, $h_2 = H_2 = \text{const}$, and $u_{1,2} = U_{1,2}(y)$ gives

$$
\begin{aligned}
\partial_t u_2 + (U_2' - f) v_2 - \frac{F}{2}\partial_y v_2 H_2 &= 0, && (18)\\
\partial_t v_2 + f u_2 - \frac{F}{2}\partial_y u_2 H_2 + \frac{F}{2}U_2'\eta &= -\partial_y \phi, && (19)\\
\partial_t u_1 + (U_1' - f) v_1 - \frac{F}{2}\partial_y v_1 H_1 &= 0, && (20)\\
\partial v_1 + f u_1 - \frac{F}{2}\left(\partial_y u_1 H_1 + U_1'\eta + 2\partial_y \eta U_1\right) &= -\partial_y\phi - g'\partial_y\eta + F\partial_y\left(-\eta U_2 + H_2\partial_y u_2\right) && (21)\\
\pm\partial_t \eta_{1,2} + \partial_y v_{1,2} H_{1,2} &= 0, && (22)
\end{aligned}
$$

where we used the weak stratification approximation, and introduced the geopotential $\phi = \pi/\rho$, the perturbation of the interface position η, and the prime notation for ordinary derivatives. Equations (22) suggest introduction of a new variable $V = v_1 H_1 = -v_2 H_2$. In terms of this variable we get:

$$
\begin{aligned}
\partial_t V + f H_1 u_1 &= -H_1\partial_y\phi - g' H_1 \partial_y \eta\\
&+ F H_1\left[\frac{H_1}{2}\partial_y u_1 + \frac{U_1'}{2}\eta + U_1\partial_y\eta + H_2\partial_y u_2 - \partial_y\left(U_2\eta\right)\right], && (23)\\
-\partial_t V + f H_2 u_2 &= -H_2\partial_y\phi + F H_1\left(\frac{H_2}{2}\partial_y u_2 - \frac{U_2'}{2}\eta\right), && (24)\\
H_1\partial_t u_1 &= -\left(U_1' - f\right)V + \frac{F}{2}H_1\partial_y V, && (25)\\
H_2\partial_t u_2 &= +\left(U_2' - f\right)V - \frac{F}{2}H_2\partial_y V. && (26)
\end{aligned}
$$

The variable η can be eliminated from the equations (23), (24) by time-differentiation and the use of (22), and then the variables u_1 and u_2 can be expressed in terms of V using (25), (26). Finally, geopotential can be eliminated by suitably combining the remaining equations. We arrive in this way to a single equation for V. The whole procedure follows that of Zeitlin (2008) for traditional approximation, but new terms appear in the final equation due to NT contributions. We make a Fourier transformation in time: $V(t,y) = e^{i\omega t}\hat{V}(\omega,y) + \text{c.c}$ and arrive to a second-order ordinary differential equation for $\hat{V}$, which we write down in the case $H_1 = H_2 = H$, for simplicity:

$$\begin{aligned}\left[-2\omega^2 + f(f - U_1'(y)) + f(f - U_2'(y)) + \frac{FH}{2}\left(U_1''(y) + U_2''(y)\right)\right]\hat{V}(y) &\quad + \\ -HFU_1'(y)\hat{V}'(y) - H\left[g' + FH(U_2(y) - U_1(y)) + \frac{F^2H^2}{2}\right]\hat{V}''(y) &= 0. \quad (27)\end{aligned}$$

2.4 Trapped modes and inertial instability

We introduce the length scale L, the time scale f^{-1}, the velocity scale U and non-dimensional parameters: Rossby number $Ro = U/(fL)$, Burger number $Bu = g'H/(f^2L^2)$, and $\delta_{\mathrm{NT}} = \frac{FH}{fL}$ and write down the equation (27) in non-dimensional form:

$$\begin{aligned}\left[2(1-\omega^2) - Ro\,(U_1'(y)) + U_2'(y)) + Ro\,\delta_{\mathrm{NT}}\frac{(U_1''(y) + U_2''(y))}{2}\right]\hat{V}(y) &\quad + \\ \delta_{\mathrm{NT}}RoU_1'(y)\hat{V}'(y) - \left[Bu + \delta_{\mathrm{NT}}\left((U_2(y) - U_1(y))\right) + \frac{\delta_{\mathrm{NT}}^2}{2}\right]\hat{V}''(y) &= 0. \quad (28)\end{aligned}$$

We can eliminate the term with the first derivative by the transformation of the dependent variable:

$$\hat{V} \to \hat{V}e^{\delta_{\mathrm{NT}}Ro\frac{U_1'}{2\left[Bu+\delta_{\mathrm{NT}}(U_2-U_1)+\frac{\delta_{\mathrm{NT}}^2}{2}\right]}}. \quad (29)$$

We should recall that, by construction, in the rotating shallow water model the vertical scale H is much smaller than the typical horizontal scale L, and hence the parameter δ_{NT} is necessarily small. We will write down the equation resulting from the substitution (29) into (28) in the leading order in δ_{NT}:

$$\begin{aligned}\left[Bu + \delta_{\mathrm{NT}}\left((U_2(y) - U_1(y))\right)\right]\hat{V}'' &\quad + \\ \left[2\omega^2 - \left[2 - Ro\,(U_1'(y)) + U_2'(y)) + Ro\,\delta_{\mathrm{NT}}\frac{(U_1''(y) + U_2''(y))}{2}\right]\right]\hat{V} &= 0. \quad (30)\end{aligned}$$

This equation is a Scrödinger equation with energy $2\omega^2$ and "potential" $2 - Ro\,(U_1' + U_2') + Ro\,\delta_{\mathrm{NT}}\frac{(U_1''+U_2'')}{2}$. Under traditional approximation, when $\delta_{\mathrm{NT}} \to 0$, $U_1 = U_2 = U$, cf (11), (12) at constant h_1. Equation (30) then has negative eigenvalues ω^2 for strong enough anticyclonic ($U'(y) > 0$) shears of the background flow (Zeitlin, 2008), and hence gives an instability of standing modes trapped in the minimum of the potential. In the presence of NT effects the background flow acquires a vertical shear. As follows from (11), (12) at constant h_1, in non-dimensional terms

$$U_1 = U + \delta_{\mathrm{NT}}\left(U + U'/2\right), \quad U_2 = U + \delta_{\mathrm{NT}}U'/2, \quad (31)$$

and in the leading order in δ_{NT} (30) gives

$$\frac{Bu}{2}\hat{V}''(y) + \left[\omega^2 - (1 - Ro(1+\delta_{\mathrm{NT}})U'(y))\right]\hat{V}(y) = 0. \quad (32)$$

Therefore, at small δ_{NT} the influence of NT effects to the leading order consists just in deepening the potential by rescaling its amplitude, and hence in diminishing eigenfrequencies squared, which leads to increase of the growth rates of the instability. At the same time, the eigensolutions $\hat{V}(y)$ are distorted with respect to the traditional approximation, according to (29).

3 Inertial instability in continuously stratified fluid on the non-traditional f- plane

3.1 Scaling, parameters and background flow

In this section we are back to the full primitive equations (1) - (5). For simplicity, we consider a rest state with a stable linear stratification ρ_s and constant Brünt-Vaïsaïla frequency $N = \sqrt{-g\frac{\partial_z \rho_s}{\rho_0}}$. We introduce a horizontal velocity scale U, a horizontal scale L, a geopotential scale $\phi_0 = gH$, and time scale f^{-1}. The dimensionless parameters are then the Rossby number $Ro = U/(fL)$, the baroclinic Burger number $Bu = (NH/(fL))^2$, and a parameter $\gamma = F/f$ measuring the magnitude of NT terms. As in Tort et al. (2016), we consider a reference barotropic balanced shear flow $(\Phi_{\mathrm{ref}}(y,z), u_{\mathrm{ref}}(y,z))$ with the background vertical structure defined by $(\rho_s(z), \Phi_s(y,z))$. This flow verify the equations:

$$f u_{\mathrm{ref}} + \partial_y \Phi_{\mathrm{ref}} = 0, \quad \partial_z \Phi_{\mathrm{ref}} + b_0 - F U_{\mathrm{ref}} = 0, \tag{33}$$

where $b_0 = g\rho_s/\rho_0$. By cross-differentiation we get:

$$(f\partial_z + F\partial_y)\, u_{\mathrm{ref}} = 0, \tag{34}$$

Hence, the velocity of the barotropic flow is a function of both meridional and vertical coordinates which, in turn, means that linearization about such profile gives a non-separable problem. However, using the crucial observation that the flow velocity is a function of "slanted" meridional coordinate $\xi = y - (F/f)z$ only: $u_{\mathrm{ref}}(y,z) = U(\xi)$, and making a change of variable $y \to \xi$, we get $\partial_y \Phi_{\mathrm{ref}} = \partial_\xi \Phi$, $\partial_z \Phi_{\mathrm{ref}} = \partial_z \Phi_0(z) - F/f \partial_\xi \Phi$, where $\Phi_{\mathrm{ref}}(y,z) = \Phi(\xi) + \Phi_s(z)$. Using (33) we arrive to the standard thermal wind balance in terms of (ξ, z):

$$fU + \partial_\xi \Phi = 0, \quad \partial_z \Phi_0 + b_0 = 0. \tag{35}$$

3.2 Reduction to 2.5 dimensions, linearization, and resulting eigenfrequency problem

As in section 3, we are interested in a zonally symmetric problem where all dependence on x is removed in (1) - (5). We thus get a "2.5 dimensional" version of equations (1) - (5) which, after linearization about the flow given by (35) become:

$$\partial_t u - (f - U'(\xi))\hat{v} = 0, \tag{36}$$
$$\partial_t (\hat{v} + \gamma w) + f' + \partial_\xi \phi = 0, \tag{37}$$
$$\partial_t b - N^2 w = 0, \tag{38}$$
$$\delta_{\mathrm{NH}} \partial_t w + (\partial_z - \gamma \partial_\xi)\phi + b - Fu = 0, \tag{39}$$
$$\partial_\xi \hat{v} + \partial_z w = 0, \tag{40}$$

where all dependent variables represent small perturbations of the reference state, and we introduced a new variable $\hat{v} = v - \gamma w$. By resolving the incompressibility constraint (40) with the help of streamfunction $\psi(\xi, z, t)$: $\hat{v} = -\partial_z \psi$, $w = \partial_\xi \psi$ and eliminating all variables in favor of ψ we get:

$$\partial^4_{zztt}\psi + \left(\delta_{\mathrm{NH}} + \gamma^2\right)\partial^4_{\xi\xi tt}\psi - 2\gamma \partial^4_{\xi ztt}\psi + f\left(f - U'(\xi)\right)\partial^2_{zz}\psi + N^2 \partial^2_{\xi\xi}\psi = 0. \tag{41}$$

Under the above-described scaling, the non-dimensional form of this equation is:

$$\partial^4_{zztt}\psi + \frac{H^2}{L^2}\left(\delta_{\mathrm{NH}} + \gamma^2\right)\partial^4_{\xi\xi tt}\psi - 2\gamma\frac{H}{L}\partial^4_{\xi ztt}\psi + \left(1 - Ro U'(\xi)\right)\partial^2_{zz}\psi + Bu\,\partial^2_{\xi\xi}\psi = 0. \tag{42}$$

After the Fourier transformation $\psi = \hat{\psi}(\xi)e^{i(mz-\omega t)}$ + c.c. this gives an eigenproblem for eigenfrequencies ω:

$$(1 - \omega^2 \left(\delta_{\rm NH}^2 + \delta_{\rm NT}^2\right) \hat{\psi}''(\xi) + 2\omega^2 \delta_{\rm NT}\, im\, \hat{\psi}'(\xi) + m^2 \left[\omega^2 - (1 - Ro\, U'(\xi))\right] \hat{\psi}(\xi) = 0, \quad (43)$$

where we put $Bu = 1$ without loss of generality (results below can be extended to any Bu by rescaling), transformed the hydrostatic switch into the true non-hydrostaticity parameter $\delta_{\rm NH} = \frac{H}{L}$, and introduced the parameter $\delta_{\rm NH} = \gamma \frac{H}{L}$already used in sect. 2. It is, in fact, this parameter, and not γ which measures the strength of NT effects, cf Tort et al. (2016). As in section 3 we arrive to a second-order ordinary differential equation, where the term with the first derivative in ξ can be removed by a change of variables

$$\hat{\psi} = \tilde{\psi} e^{-im\delta_{\rm NT} \frac{\omega^2}{1-\omega^2(\delta_{\rm NH}^2+\delta_{\rm NT}^2)}}. \quad (44)$$

We thus get:

$$\tilde{\psi}''(\xi) + \frac{m^2}{1 - \omega^2 \left(\delta_{\rm NH}^2 + \delta_{\rm NT}^2\right)} \left[\omega^2 \left(1 + \delta_{\rm NT}^2 \frac{\omega^2}{1 - \omega^2 \left(\delta_{\rm NH}^2 + \delta_{\rm NT}^2\right)}\right) - (1 - RoU'(\xi))\right] \tilde{\psi}(\xi) = 0. \quad (45)$$

3.3 Analysis of the eigenproblem, trapped modes and inertial instability

The eigenproblem (45) has the same form as in the traditional approximation, cf Plougonven and Zeitlin (2009), modulo the change $\delta_{\rm NH}^2 \to \delta_{\rm NH}^2 + \delta_{\rm NT}^2$ in the factor in front of the square bracket, and the replacement $\omega^2 \to \omega^2 \left(1 + \delta_{\rm NT}^2 \frac{\omega^2}{1 - \omega^2 \left(\delta_{\rm NH}^2 + \delta_{\rm NT}^2\right)}\right)$ in the first term inside the square brackets. The trapped eigensolutions of (45), which can be interpreted again as a Schrödinger equation, appear when the "potential" (the second in square brackets term) is positive, i.e. the shear of $U(\xi)$ is anticyclonic. When the "potential" is deep enough, the eigenvalues, i.e. the eigenfrequencies squared, can become negative, thus giving an instability. This analysis is the same as in traditional approximation (Plougonven and Zeitlin, 2009), and similar to that of section 2. The above-described changes due to NT effects increase the effective non-hydrostaticity of the flow, and enhance the growth rates, respectively. Note that the changes in the eigenproblem (45) are of the order $\delta_{\rm NT}^2$. If this parameter is small, which is the case for large-scale geophysical flows, the NT effects in the leading order just change the form of eigensolutions, as follows from (44) together with the change of variables $\hat{\psi}(y) \to \hat{\psi}(\xi)$. At the next order they increase the growth rates. As was shown in (Plougonven and Zeitlin, 2009) the eigenproblem (45) can be solved analytically in the case of *tanh* profile of velocity. By making the above-described changes in the parameters of the resulting hypergeometric equation, the dependence of the eigenfrequencies squared on the parameter $\delta_{\rm NT}$ can be explicitly traced. In particular, in the hydrostatic approximation $\delta_{\rm NH} = 0$ at small $\delta_{\rm NT}$, it can be shown that the NT correction to the eigenfrequency squared is strictly negative, and thus leads to the increase of the growth rate of inertial instability.

4 Conclusions and discussion

We thus showed how the NT effects change the Schrödinger-like equation for the waves trapped in the anticyclonic shear, which are at the origin of the inertial instability, both in the two-layer and continuously stratified models. The analysis of the simplest barotropic configuration of the background flow can be done "by hand" and gives qualitative understanding of the role of NT effects. We should emphasize that the use of "slanted" variables, as in section 3, allows to analyze analytically the role of NT effects upon other dynamical problems in simple flow configurations., like for example frontogenesis, following the approach of Plougonven and Zeitlin (2005). This will be presented elsewhere.

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