A new algorithm of evaluation of Wigner Distribution Function with application to radio occultation analysis

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Introduction

Introduction to Fractional Fourier Transform

The Fractional Fourier Transform is a generalization of the Fourier Transform, which allows for the rotation of signals in the time-frequency domain. It is defined as:

\[ F^\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha \theta} \hat{f}(\theta) e^{i\theta x} d\theta \]

where \( \alpha \) is the fractional angular parameter and \( \hat{f}(\theta) \) is the Fourier transform of the function \( f(x) \).

Wigner and Kirkwood Distribution Functions

The Wigner Distribution Function (WDF) is defined as:

\[ W(x, t) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^{i2\pi x \gamma} \left[ \hat{f}(\gamma) \delta(\gamma - t) + \frac{1}{\sigma^2} \hat{f}(\gamma) \delta(\gamma - t) \right] d\gamma \]

where \( \hat{f}(\gamma) \) is the Fourier transform of \( f(t) \). The Kirkwood Distribution Function (KDF) is a simpler form of the WDF, defined as:

\[ K(x, t) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} e^{i2\pi x \gamma} \left[ \hat{f}(\gamma) \delta(\gamma - t) \right] d\gamma \]

Fractional Fourier Transform

Consider a phase space rotation \( (x, t) \rightarrow (x', t') \) by angle \( \theta \):

\[ \theta = \frac{\pi}{2} \alpha \]

where \( \alpha \) is the fractional angular parameter.

Properties of the Fractional Fourier Transform:

1. The Fractional Fourier Transform is a linear operation.
2. The Fractional Fourier Transform is a unitary operation.
3. The Fractional Fourier Transform is a time-frequency representation.
4. The Fractional Fourier Transform is a rotation in the time-frequency plane.

Examples of KDF for Test Signals Subjected to FFT

The evaluation of KDF is computationally cheap as compared to WDF: WDF requires a large number of long-term Fourier transform; KDF requires just one. Using the properties of the FrFT and the convolution relations between WDF and KDF, we can write the following relations:

\[ \hat{K}(\theta) \sin(\theta \alpha) = \hat{W}(\theta) \]

where \( \hat{K}(\theta) \) is the Fourier transform of the KDF, and \( \hat{W}(\theta) \) is the Fourier transform of the WDF.

The Evaluation of Smoothed WDF from KDF

The evaluation of \( \omega \) is computationally cheap as compared to WDF: WDF requires a large number of long-term Fourier transform; KDF requires just one. Using the properties of the FrFT and the convolution relations between WDF and KDF, we can write the following relations:

\[ \hat{W}(\theta) \sin(\theta \alpha) = \hat{K}(\theta) \]

Conclusions

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